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Supersymmetry reduction of N-extended supergravities in four dimensions

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Abstract: We consider the possible consistent truncation of N-extended supergravities to lower N' theories. The truncation, unlike the case of N-extended rigid theories, is non trivial and only in some cases it is sufficient just to delete the extra $N - N'$ gravitino multiplets. We explore different cases (starting with $N = 8$ down to $N' \geq 2$) where the reduction implies restrictions on the matter sector. We perform a detailed analysis of the interesting case $N = 2 \longrightarrow N = 1$. This analysis finds applications in different contexts of superstring and M-theory dynamics.

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Contents

1. Introduction

It is well known that for globally supersymmetric theories, with particle content of spin $0, 1/2, 1$ any theory with N supersymmetries can be regarded as a particular case of a theory with a number $N' < N$ of supersymmetries [1]. To prove this it is sufficient to decompose the N supersymmetry-extended multiplets into N'-multiplets.

JHEP03(2002)025

Of course N-extended supersymmetry is more restrictive than $N' < N$ supersymmetry implying that the former will only allow some restricted couplings of the latter. As we are going to show in the present paper the same argument does not apply to supergravity theories. Indeed, let us consider a standard N-extended supergravity theory with N gravitini and a given number of matter multiplets with spin $0, 1/2, 1$: then the N'-extended supergravity obtained by reduction from the mother theory will no longer be standard because a certain number $N - N'$ of spin $3/2$ multiplets appear in the decomposition. Therefore to obtain a standard N'-extended supergravity one must truncate out at least the $N - N'$ spin $3/2$ multiplets and all the non-linear couplings they generate in the supergravity action.

The most known example is $N = 2$ supergravity in presence of hypermatter [2]–[5]. The non-linear couplings of the hypermultiplets generate what is called a "quaternionic geometry" [2]. If we regard the $N = 2$ hypermultiplets as a pair of $N = 1$ Wess-Zumino multiplets, what we obtain is incompatible with $N = 1$ supergravity where the non-linear couplings must describe a Kähler-Hodge manifold geometry [6]. Therefore, in order to consistently reduce a $N = 2$ supergravity to a $N = 1$ theory, the former theory must have the property that a certain submanifold of the original quaternionic manifold be a Kähler-Hodge manifold.

Note that in rigid supersymmetry hypermultiplet couplings are described by HyperKähler geometry which is instead compatible with $N = 1$ supersymmetry.

As an illustrative example let us consider maximal $N = 8$ supergravity in $D = 4$ [7] truncated to lower N' supergravities. In this situation the consistent truncation consists in deleting only spin $3/2$ multiplets for sufficiently high N' ($N' = 6, 5, 4$), but for $N' \leq 4$, where the matter sectors begin to appear, the consistent truncation also requires to delete some matter multiplets. We will illustrate how this process of reduction can be understood in group-theoretical and geometrical terms, by requiring that certain geometrical conditions dictated by supergravity define some submanifold of the original scalar manifold $E_{7(7)}/\,\mathrm{SU}(8)$ of $N = 8$ supergravity.

Returning to the $N = 2 \longrightarrow N = 1$ case, we can show that this generally demands a reduction of both special Kähler manifold ($\mathcal{M}^{SK}(n_V)$) [3, 8, 9, 5] and quaternionic manifold ($\mathcal{M}^{Q}(n_H)$), where n_V and n_H are the number of vector multiplets and hypermultiplets respectively. By equipping these manifolds with complex coordinates $z^{\mathcal{I}} \in \mathcal{M}^{SK}$ ($\mathcal{I} = 1, \dots, n_V$) and real coordinates $q^u \in \mathcal{M}^Q$ ($u = 1, \dots, 4n_H$) the Riemann tensors are given respectively by:

$$\begin{aligned} R_{\mathcal{I}\overline{\mathcal{J}}\mathcal{K}\overline{\mathcal{L}}} &= g_{\mathcal{I}\overline{\mathcal{L}}}g_{\mathcal{K}\overline{\mathcal{J}}} + g_{\mathcal{K}\overline{\mathcal{L}}}g_{\mathcal{I}\overline{\mathcal{J}}} - \overline{C}_{\overline{\mathcal{J}}\overline{\mathcal{L}}\overline{\mathcal{N}}}C_{\mathcal{I}\mathcal{K}\mathcal{M}}g^{\mathcal{M}\overline{\mathcal{N}}} \\ \mathcal{R}^{uv}{}_{pq}\mathcal{U}^{\alpha A}_u\mathcal{U}^{\beta B}_v &= -\frac{\mathrm{i}}{2}\Omega^x_{pq}(\sigma_x)^{AB}C^{\alpha\beta} + \mathbb{R}^{\alpha\beta}_{pq}\,\epsilon^{AB}\,, \end{aligned} \qquad (1.1)$$

where the SU(2) triplet and singlet parts are the SU(2) and $\mathrm{Sp}(2n_H)$ curvatures respectively.[1] Here $A, B = 1, 2; x = 1, 2, 3$ are indices of the fundamental and adjoint representation of SU(2) and $\alpha, \beta, \ldots = 1, \ldots, 2n_H$ are indices in the fundamental representation of $\mathrm{Sp}(2n_H)$.

[1]Here by $\mathrm{Sp}(2n_H)$ we denote the compact form of the symplectic group sometimes called $\mathrm{USp}(2n_H)$ (i.e. $\mathrm{Sp}(2) = \mathrm{SU}(2)$).

JHEP03(2002)025

The Kähler metric $g_{\mathcal{I}\overline{\mathcal{L}}} = \partial_{\mathcal{I}}\partial_{\overline{\mathcal{L}}}\mathcal{K}$, with $\mathcal{K} = -\log[\mathrm{i}(\overline{X}^{\mathbf{\Lambda}} F_{\mathbf{\Lambda}} - \overline{F}_{\mathbf{\Lambda}} X^{\mathbf{\Lambda}})]$, is given in terms of $(X^{\mathbf{\Lambda}}, F_{\mathbf{\Lambda}})$ which are the holomorphic symplectic sections of $\mathcal{M}^{SK}$ (they are related to the covariantly holomrphic symplectic sections $(L^{\mathbf{\Lambda}}, M_{\mathbf{\Lambda}})$ by $(L^{\mathbf{\Lambda}}, M_{\mathbf{\Lambda}}) = e^{\mathcal{K}/2}(X^{\mathbf{\Lambda}}, F_{\mathbf{\Lambda}})$). The tensor $C_{\mathcal{IKM}}$ is threefold symmetric and covarianty holomorphic, i.e. $C_{\mathcal{IKM}} = e^{\mathcal{K}} W_{\mathcal{IKM}}$ (with $W_{\mathcal{IKM}}$ holomorphic).

On $\mathcal{M}^Q$, $\mathcal{U}^{\alpha A}$ denotes the vielbein 1-form. Furthermore, we have:

$$\begin{aligned}
\Omega^x &\equiv d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z = -\mathrm{i}\,\mathbb{C}_{\alpha\beta}(\sigma_x)_{AB}\mathcal{U}^{\alpha A} \wedge \mathcal{U}^{\beta B}, \\
\mathbb{R}^{\alpha}{}_{\beta} &\equiv d\Delta^{\alpha}{}_{\beta} + \Delta^{\alpha}{}_{\gamma} \wedge \Delta^{\gamma}{}_{\beta} \\
&= -\epsilon_{AB}\mathcal{U}^{A\alpha} \wedge \mathcal{U}^{B}_{\beta} + \mathcal{U}^{A\gamma} \wedge \mathcal{U}^{B\delta}\epsilon_{AB}\mathbb{C}^{\alpha\rho}\Omega_{\rho\beta\gamma\delta}\,,
\end{aligned} \tag{1.2}$$

where $\Omega_{\rho\beta\gamma\delta}$ is completely symmetric in its four indices.

The $N = 2 \to N = 1$ reduction imposes a number of conditions on the above defined structures, which have to be satisfied in order to have a consistent reduction. In particular, we find that the two scalar manifolds $\mathcal{M}^{SK}$ and $\mathcal{M}^Q$ have to be reduced to the submanifolds $\mathcal{M}_R(n_C) \subset \mathcal{M}^{SK}$ and $\mathcal{M}^{KH}(n_h) \subset \mathcal{M}^Q$, where $n_C \leq n_V - n'_V$, $n_h \leq n_H$ are the complex dimensions of the two Kähler–Hodge manifolds $\mathcal{M}_R$ and $\mathcal{M}^{KH}$ and n'_V is the number of $N = 1$ vector multiplets.

We first discuss the two extreme cases $n'_V = n_V$ ($n_C = 0$) and $n'_V = 0$ ($n_C = n_V$). In the first case no $N = 1$ chiral multiplet coming from $N = 2$ vector multiplet is retained and all $N = 1$ vector multiplets may remain. In the second case, all the $N = 1$ vector multiplets are truncated out, and no restrictions appear on the special-Kähler manifold: $\mathcal{M}_R = \mathcal{M}^{SK}$.

In the general case, let us decompose the coordinates on $\mathcal{M}^{SK}$:

$$z^{\mathcal{I}} \to (z^i, z^{\alpha}) \tag{1.3}$$

and those on $\mathcal{M}^Q$:

$$q^u \to (w^s, \overline{w}^{\overline{s}}, n^t, \overline{n}^{\overline{t}})\,, \tag{1.4}$$

where z^i ($i = 1, \ldots, n_C$) and w^s ($s = 1, \ldots, n_h$) are the holomorphic coordinates in $\mathcal{M}_R$ and $\mathcal{M}^{KH}$ respectively, and z^{α} ($\alpha = 1, \ldots, n'_V = n_V - n_C$) and n^t ($t = 1, \ldots, n_H - n_h$) are the holomorphic coordinates in their orthogonal complements. Splitting furthermore the $N = 2$ vector indices $\mathbf{\Lambda} \to (\Lambda, X)$, where $\Lambda = 1, \ldots, n'_V$ and $X = 0, 1, \ldots, n_C$, we find the following constraints to be satisfied on $\mathcal{M}_R \times \mathcal{M}^{KH}$ from supersymmetry reduction. On $\mathcal{M}_R$ we get, for consistent reduction of the special geometry sector in the ungauged case:

$$\begin{aligned}
C_{ij\alpha}|_{\mathcal{M}_R} &= 0\,; & C_{\alpha\beta\gamma}|_{\mathcal{M}_R} &= 0 \\
L^{\Lambda}|_{\mathcal{M}_R} &= 0\,, & f_i^{\Lambda}|_{\mathcal{M}_R} &\equiv \nabla_i L^{\Lambda}|_{\mathcal{M}_R} = 0 \\
f_{\alpha}^{X}|_{\mathcal{M}_R} &\equiv \nabla_{\alpha} L^{X}|_{\mathcal{M}_R} = 0\,.
\end{aligned} \tag{1.5}$$

The parent (non holomorphic) vector kinetic matrix $\mathcal{N}_{\mathbf{\Lambda\Sigma}}$ satisfies on $\mathcal{M}_R$:

$$\mathcal{N}_{\Lambda Y}|_{\mathcal{M}_R} = 0\,. \tag{1.6}$$

JHEP03(2002)025

Furthermore we obtain that $\overline{\mathcal{N}}_{\Lambda\Sigma}|_{\mathcal{M}_R} \equiv \frac{1}{2} f_{\Lambda\Sigma}$ is holomorphic, while $\mathcal{N}_{XY}$ has no restrictions and gives the period matrix on $\mathcal{M}_R$, which is indeed a Special-Kähler manifold.

For the hypermultiplet sector, the reduction is more subtle because we have to reduce the holonomy from $\mathrm{SU}(2)\times\mathrm{Sp}(2n_H)$ to $\mathrm{U}(1)\times\mathrm{SU}(n_h)$ which corresponds to decompose the SU(2) indices $A, B, \ldots, \to (1,2)$ and the $\mathrm{Sp}(2n_H)$ indices $\alpha, \beta \ldots \to (I, \dot{I})$. The following constraints are found on the geometrical structure of the manifold $\mathcal{M}^{KH} \subset \mathcal{M}^Q$

$$\begin{aligned} \Omega_{IJK\dot{L}}|_{\mathcal{M}^{KH}} &= 0 \\ \mathcal{U}^{2I}|_{\mathcal{M}^{KH}} &= (\mathcal{U}^{1\dot{I}})^*|_{\mathcal{M}^{KH}} = 0\,. \end{aligned} \tag{1.7}$$

In particular, the second equation implies that the complex scalars of the chiral multiplets coming from the reduced quaternionic manifold are at most half of the quaternionic dimension of the original $N = 2$ manifold [10].

The present investigation concerning the $N = 2 \to N = 1$ reduction is further analyzed in the most general case when isometries of the scalar manifolds are gauged.

In particular we find that the number of reduced $N = 1$ vector multiplets and of $N = 1$ chiral multiplets obtained by truncation of the $N = 2$ vector multiplets (which are in the adjoint representation of some gauge group $G^{(2)}$) depend on the gauge group $G^{(1)}$ under which the reduced hypermultiplets are charged. Indeed, if $Adj(G^{(2)}) \to Adj(G^{(1)}) + R(G^{(1)})$, then the chiral multiplets coming from $N = 2$ vector multiplets are in $R(G^{(1)})$.

The reduction of the gauge group further implies constraints on the special geometry and quaternionic Killing vectors and prepotentials [11, 5]. For the kählerian Killing vectors $k^{\mathcal{I}}_{\mathbf{\Lambda}}$ and prepotential $P^0_{\mathbf{\Lambda}}$ we find:

$$\begin{aligned} k^i_X &= 0\,, \quad k^\alpha_\Lambda = 0 \\ k^i_\Lambda &= \mathrm{i} g^{i\bar{\jmath}} \partial_{\bar{\jmath}} P^0_\Lambda \neq 0 \\ P^0_X &= 0\,. \end{aligned} \tag{1.8}$$

Furthermore, for the quaternionic Killing vectors $k^u_{\mathbf{\Lambda}}$ and SU(2)-valued prepotentials $P^x_{\mathbf{\Lambda}}$, we find:

$$\begin{aligned} k^s_X &= 0\,, \quad k^t_\Lambda = 0\,, \\ k^s_\Lambda &= \mathrm{i} g^{s\bar{s}} \partial_{\bar{s}} P^3_\Lambda \neq 0\,, \\ P^3_X &= 0\,, \\ P^{\mathrm{i}}_\Lambda &= 0\,, \quad (\mathrm{i} = 1,2)\,. \end{aligned} \tag{1.9}$$

The $N = 1$ D-term and superpotential are respectively given by:[2]

$$\begin{aligned} D^\Lambda &= -2(\mathrm{Im} f)^{-1\Lambda\Sigma}(P^0_\Sigma(z,\bar{z}) + P^3_\Sigma(w,\bar{w})) \\ L &= e^{\frac{\mathcal{K}_R+\mathcal{K}_H}{2}} W(z,w) = \frac{\mathrm{i}}{2} L^X \left(P^1_X - \mathrm{i} P^2_X\right), \end{aligned} \tag{1.10}$$

where $\mathcal{K}_R$, $\mathcal{K}_H$ are the Kähler potentials on $\mathcal{M}_R$ and $\mathcal{M}^{KH}$ respectively.

[2]Particular cases of these formulae have been obtained in [12]–[22].

JHEP03(2002)025

This reduction may find applications and is in fact related to many interesting aspects of string theory or M theory compactified on a Calabi-Yau threefold. Indeed M-theory on a Calabi-Yau threefold originates a $N=2$ theory in five dimensions [23]. Trivial reduction on S^1 would give a $N=2$ theory in $D=4$. However, if we reduce on the orbifold S^1/Z_2 [24]–[28], then we obtain a $N=1$ theory with a particular truncation of the $D=5, N=2$ supergravity states. Other applications are related to brane-dynamics where the theory on the brane has lower supersymmetry than the theory on the bulk [29, 30].

A different mechanism is obtained by considering type IIB theory on a Calaby-Yau threefold in presence of H-fluxes [12]–[22] where also $N=1$ (or $N=0$) supersymmetric vacua can be studied.

A related issue is the partial supersymmetry breaking of $N=2$ down to $N=1$ through a superHiggs mechanism [31, 32]. If one integrates out the massive gravitino, then the theory should become a $N=1$ theory. In this case to "integrate out" is in principle different from truncating unless very special situations occur. However in the minimal model studied in reference [31], the resulting $N=1$ lagrangian is a particular case of the general case studied here.

The paper is organized as follows: in section 2 we study the decomposition of the $N=8$ supergravity multiplet into $N'<8$ supermultiplets and infer the reduced theories from group-theoretical arguments.

In section 3 we extend the analysis to three, five and six dimensional maximal supergravities reduced to eight supercharges.

In section 4 we give the interpretation of the reduction procedure in a geometrical setting which will be useful to apply our results to the specific problem of the $N=2 \longrightarrow N=1$ reduction.

In section 5 we discuss the constraints coming from supersymmetry when the reduction procedure is applied to ungauged theories.

In section 6, which is the heart of the paper, we give the analysis of the $N=2 \longrightarrow N=1$ reduction in detail, also in presence of gauging, both in the vector multiplet and hypermultiplet sectors. At the beginning of the section we discuss the constraints coming from the gravitino truncation, while in section 6.1 and 6.2 we study the reduction of the $N=2$ vector multiplet sector. Subsection 6.3 is devoted to the truncation of the hypermultiplets sector, while subsection 6.4 discusses further consequences of the gauging. In subsection 6.5 the computation of the reduction of the scalar potential is given, and finally in subsection 6.6 we give examples of supergravity models which realize this consistent truncation.

The appendices include some technical details related to the reduction. In particular, in appendices A and B we show the consistency of the $N=8 \rightarrow N=N'$ truncation in the superspace Bianchi identities formalism and we apply it to the $N=2 \rightarrow N=1$ reduction of gauged supergravity. In appendix C we prove a formula valid for the $N=2$ vector multiplets which is useful for the truncation. Appendix D refers to the reduction of the special-Kähler manifolds with special coordinates; appendix E contains the reduction of an important relation valid on quaternionic geometry in presence of isometries and appendix F shows the consistency of the reduction of the $N=2$ scalar potential to $N=1$ and exploits

JHEP03(2002)025

some magic properties of the supersymmetry Ward identities. Finally, appendix G contains the explicit form of the $N = 2$ and $N = 1$ lagrangians which are left invariant under the supersymmetry transformation laws given in the text.

2. $N = 8 \longrightarrow N'$ reduction without gauging

Reduction of $N = 8$ supergravity to $2 \leq N' \leq 6$ offers interesting examples of consistent truncations of standard supergravity [33, 34].

We restrict our analysis to theories whose σ-models are given by symmetric spaces G/H. This includes all the theories with $N' \geq 3$ and a subset of the $N = 2$ theories. The analysis turns out to be particularly simple in all these cases.

Let us first consider $N' = 5, 6$ where the reduction only involves the graviton multiplet and $N - N'$ spin 3/2 multiplets.

In the $N = 6$ case the $N = 8$ R-symmetry group SU(8) decomposes as:

$$\mathrm{SU}(8) \to \mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{SU}(2)\,, \tag{2.1}$$

where SU(2) is the group commuting with the $N' = 6$ R-symmetry U(6). Correspondingly the $N = 8$ graviton multiplet decomposes into $N' = 6$ spin 2 and 3/2 multiplets, as follows:

$$\begin{aligned}
&\left[(\mathbf{2}), 8\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 28(\mathbf{1}), 56\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 70(\mathbf{0})\right] \longrightarrow \\
&\longrightarrow \left[(\mathbf{2}), 6\left(\frac{\mathbf{3}}{\mathbf{2}}\right), (15+1)(\mathbf{1}), (20+6)\left(\frac{\mathbf{1}}{\mathbf{2}}\right), (15+\overline{15})(\mathbf{0})\right] \oplus \\
&\quad \oplus 2\left[\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 6(\mathbf{1}), 15\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 20(\mathbf{0})\right].
\end{aligned}$$

The hypersurface corresponding to freeze 40 scalars of the spin 3/2 multiplets is precisely the $N' = 6$ σ-model described by the symmetric space $\mathrm{SO}^\star(12)/\mathrm{U}(6)$. Therefore by just deleting the two spin 3/2 multiplets one obtains standard $N = 6$ supergravity.

Let us now consider $N' = 5$. In this case the decomposition of the $N = 8$ graviton multiplet into $N' = 5$ multiplets, corresponding to the R-symmetry decomposition

$$\mathrm{SU}(8) \to \mathrm{SU}(5) \times \mathrm{U}(1) \times \mathrm{SU}(3) \tag{2.2}$$

is:

$$\begin{aligned}
\left[(\mathbf{2}), 8\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 28(\mathbf{1}), 56\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 70(\mathbf{0})\right] \longrightarrow &\left[(\mathbf{2}), 5\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 10(\mathbf{1}), (10+1)\left(\frac{\mathbf{1}}{\mathbf{2}}\right), (5+\overline{5})(\mathbf{0})\right] \oplus \\
&\oplus 3\left[\left(\frac{\mathbf{3}}{\mathbf{2}}\right), (5+1)(\mathbf{1}), (10+\overline{5})\left(\frac{\mathbf{1}}{\mathbf{2}}\right), (10+\overline{10})(\mathbf{0})\right].
\end{aligned} \tag{2.3}$$

If we delete the three spin 3/2 multiplets we obtain standard $N = 5$ supergravity, or, geometrically, freezing the 60 scalars inside the spin 3/2 multiplets corresponds to single out the manifold $\mathrm{SU}(5,1)/\mathrm{U}(5) \subset E_{7(7)}/\mathrm{SU}(8)$.

JHEP03(2002)025

When $N' \leq 4$ a new phenomenon appears since in this case also matter multiplets start to appear in the decomposition of $N = 8$ supergravity into N'-extended supergravities. Therefore in this case deleting the spin 3/2 multiplets is only a necessary, but not sufficient condition to obtain a consistent N'-extended supergravity theory.

Let us first start with $N' = 4$ (this actually corresponds to compactify a type II theory in ten dimensions on $T_2 \otimes T_4/Z_2$). The decomposition of the $N = 8$ graviton multiplet into $N' = 4$ multiplets, corresponding to

$$\mathrm{SU}\,(8) \to \mathrm{SU}\,(4) \times \mathrm{SU}\,(4) \times \mathrm{U}\,(1) \tag{2.4}$$

is:

$$\begin{aligned}
\left[(\mathbf{2}), 8\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 28(\mathbf{1}), 56\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 70(\mathbf{0})\right] \longrightarrow & \left[(\mathbf{2}), 4\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 6(\mathbf{1}), 4\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 2(\mathbf{0})\right] \oplus \\
& \oplus\ 4\left[\left(\frac{\mathbf{3}}{\mathbf{2}}\right), 4(\mathbf{1}), (6+1)\left(\frac{\mathbf{1}}{\mathbf{2}}\right), (4+\overline{4})(\mathbf{0})\right] \oplus \\
& \oplus\ 6\left[(\mathbf{1}), 4\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 6(\mathbf{0})\right].
\end{aligned} \tag{2.5}$$

If we now delete the 4 spin 3/2 multiplets this is equivalent to freeze 32 scalars. When this occurs the $E_{7(7)}/\,\mathrm{SU}(8)$ manifold reduces to the submanifold $(\mathrm{SU}(1,1)/\,\mathrm{U}(1)) \times \mathrm{SO}(6,6)/\,\mathrm{SU}(4) \times \mathrm{SU}(4)$, corresponding to the product space of the $N = 4$ supergravity σ-model and the σ-model of 6 vector multiplets. In this case a standard $N' = 4$ supergravity coupled to 6 vector multiplets corresponds to a consistent truncation since $E_{7(7)} \supset \mathrm{SU}(1,1) \times \mathrm{SO}(6,6)$.

Let us now consider $N' = 3$. In this case we have the following decomposition of the $N = 8$ R-symmetry group:

$$\mathrm{SU}\,(8) \to \mathrm{SU}\,(3) \times \mathrm{U}\,(1) \times \mathrm{SU}\,(5) \tag{2.6}$$

$\mathrm{SU}\,(3) \times \mathrm{U}\,(1)$ being the R-symmetry of the $N = 3$ theory. Note that this case is dual to the $N' = 5$ case with the roles of $\mathrm{SU}(N')$ and $\mathrm{SU}(N - N')$ exchanged. The decomposition of the $N = 8$ multiplet is now:

$$\begin{aligned}
\left[(\mathbf{2}), 8(\mathbf{3/2}), 28(\mathbf{1}), 56\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 70(\mathbf{0})\right] \longrightarrow & \left[(\mathbf{2}), 3(\mathbf{3/2}), 3(\mathbf{1}), \left(\frac{\mathbf{1}}{\mathbf{2}}\right)\right] \oplus \\
& \oplus\ 5\left[(\mathbf{3/2}), 3(\mathbf{1}), 3\left(\frac{\mathbf{1}}{\mathbf{2}}\right), 2(\mathbf{0})\right] \oplus \\
& \oplus\ 10\left[(\mathbf{1}), (3+1)\left(\frac{\mathbf{1}}{\mathbf{2}}\right), (3+\overline{3})(\mathbf{0})\right].
\end{aligned} \tag{2.7}$$

If we now delete the spin 3/2 multiplet we freeze the corresponding 10 scalars. In this case, however, it is obvious that we cannot define a submanifold of $E_{7(7)}/\,\mathrm{SU}(8)$: indeed the standard $N = 3$ supergravity coupled to n vector multiplets [35] has a non linear σ-model of the form $\mathrm{SU}(3,n)/\,\mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SU}(n)$ and , for $n = 10$, $\mathrm{SU}(3,10)$ is not a subgroup of $E_{7(7)}$. Therefore we must ask the question whether there is some n for which $\mathrm{SU}(3,n) \subset E_{7(7)}$. The answer is $n = 4$ since

$$E_{7(7)} \supset \mathrm{SU}(4,4) \supset \mathrm{SU}(3,4) \times \mathrm{U}(1)\,. \tag{2.8}$$

JHEP03(2002)025

Therefore the maximal $N' = 3$ supergravity contained inside the $N = 8$ theory corresponds to the coupling with 4 matter multiplets and the corresponding σ-model lives in the submanifold

$$\mathrm{U}(3,4)/\,\mathrm{U}(3)\times\mathrm{U}(4)\subset E_{7(7)}/\,\mathrm{SU}(8)\,. \tag{2.9}$$

As far as (continuous) duality is concerned, we see that the 3 graviphotons and 4 matter vectors are in the fundamental of $\mathrm{SU}(3,4)$ as required by supersymmetry since

$$56 \to 21 + 21' + 7 + 7'\,. \tag{2.10}$$

This means that the 15+6 vectors coming from the five gravitino multiplets and six residual matter multiplets should combine in the antisymmetric of $\mathrm{SU}(3,4)$.

We note that if we instead use the chain

$$N = 8 \to N' = 4 \to N' = 3 \tag{2.11}$$

we would only obtain a non-maximal theory with three matter multiplets since in that case

$$E_{7(7)} \to \mathrm{SU}(1,1)\times\mathrm{SO}(6,6) \to \mathrm{SU}(3,3)\times\mathrm{U}(1)\,. \tag{2.12}$$

The latter is a particular case of the more general fact that $N = 4$ with $2n$ vector multiplets can be consistently truncated to $N = 3$ with n vector multiplets[3] using the chain[4]

$$\mathrm{SO}(6,2n)\supset\mathrm{SU}(3,n)\times\mathrm{U}(1) \tag{2.13}$$

and

$$\frac{\mathrm{SO}(6,2n)}{\mathrm{SO}(6)\times\mathrm{SO}(2n)}\supset\frac{\mathrm{SU}(3,n)}{\mathrm{SU}(3)\times\mathrm{SU}(n)\times\mathrm{U}(1)} \tag{2.14}$$

The last case we would like to consider is $N' = 2$ where there are two kinds of matter multiplets, namely the vector multiplets and the hypermultiplets. In the standard $N = 2$ theory the corresponding σ-model generally is not a coset, but we limit ourselves to examine this case, namely $\mathcal{M} = G/H$. The consistent truncation will now receive severe constraints on the matter content since the submanifold of the $N = 8$ σ-model must factorize as:

$$\mathcal{M}^{SK}(n_V)\times\mathcal{M}^{Q}(n_H)\subset E_{7(7)}/\,\mathrm{SU}(8)\,, \tag{2.15}$$

where we have denoted with $\mathcal{M}^{SK}(n_V)$ and $\mathcal{M}^{Q}(n_H)$ the special-Kähler and quaternionic manifolds of real dimensions $2n_V$ and $4n_H$ respectively.

The decomposition of the $N = 8$ graviton multiplet gives now:

$$\begin{aligned}\left[(\mathbf{2}),8(\mathbf{3/2}),28(\mathbf{1}),56\left(\frac{\mathbf{1}}{\mathbf{2}}\right),70(\mathbf{0})\right] \longrightarrow{}& [(\mathbf{2}),2(\mathbf{3/2}),(\mathbf{1})]\oplus\ 6\left[(\mathbf{3/2}),2(\mathbf{1}),\left(\frac{\mathbf{1}}{\mathbf{2}}\right)\right]\\ &\oplus 15\left[(\mathbf{1}),2\left(\frac{\mathbf{1}}{\mathbf{2}}\right),2(\mathbf{0})\right]\oplus 20\left[\left(\frac{\mathbf{1}}{\mathbf{2}}\right),2(\mathbf{0})\right].\end{aligned} \tag{2.16}$$

[3] Note that in string theory this would imply $n = 11$ in agreement with [36]

[4] $N = 3$ models based on brane flux supersymmetry breaking have recently been constructed [37].

JHEP03(2002)025

We immediately see that deleting the spin 3/2 multiplets all the scalars survive. So the question is now, how many scalars we must delete so that the scalar submanifold enjoys the above property of reducing to $\mathcal{M}^{SK}(n_V) \times \mathcal{M}^Q(n_H)$.

Two immediate solutions are obtained [38]. For $n_H = 0$, $n_V = 15$ we find:

$$\mathcal{M}^{SK}(n_V = 15) = \overset{*}{\mathrm{SO}}(12)/U(6) \subset E_{7(7)}/\,\mathrm{SU}(8) \tag{2.17}$$

which is indeed a special-Kähler manifold (coinciding with the σ-model of $N = 6$ supergravity). The other solution is $n_V = 0$, $n_H = 10$ for which

$$\mathcal{M}^Q(n_H = 10) \;=\; E_{6(2)}/\,\mathrm{SU}(6) \times \mathrm{SU}(2) \subset E_{7(7)}/\,\mathrm{SU}(8) \tag{2.18}$$

which is indeed a quaternionic space. It corresponds to the σ-model obtained by compactification of type IIB on T_6/Z_3 where only the untwisted states were retained.

By c-map of (2.18) we obtain another solution with $n_V = 9$ and $n_H = 1$ corresponding to type IIA on T_6/Z_3 [39]:

$$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)} \subset E_{7(7)}/\,\mathrm{SU}(8) \tag{2.19}$$

If we look for other maximal subgroups $G_1 \times G_2 \subset E_{7(7)}$ we find [40] (see table 1):

$$\mathrm{SO}(6, 2n) \;\subset\; \mathrm{SU}(3, n) \times \mathrm{U}(1)$$
$$G_1 \times G_2 \;=\; \mathrm{Sp}(6, R) \times G_{2(2)}\,; \quad \mathrm{SU}(1,1) \times F_{4(4)}\,; \quad \mathrm{SU}(1,1) \times \mathrm{SO}(6,6)\,; \quad \mathrm{SU}(4,4)\,. \tag{2.20}$$

The first two correspond to $(n_V, n_H) = (6, 2)$ and to its c-map image $(1, 7)$, namely:

$$\frac{\mathrm{Sp}(6,R)}{U(3)} \times \frac{G_{2(2)}}{\mathrm{SO}(4)} \subset E_{7(7)}/\,\mathrm{SU}(8) \tag{2.21}$$

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{F_{4(4)}}{\mathrm{Usp}(6) \times \mathrm{Usp}(2)} \subset E_{7(7)}/\,\mathrm{SU}(8) \tag{2.22}$$

From the last two cases we can obtain a $N = 2$ truncation of $N = 4$ and $N = 3$ supergravities with six and four hypermultiplets respectively:

$$\frac{\mathrm{SO}(6,6)}{\mathrm{SO}(6) \times \mathrm{SO}(6)} \longrightarrow \frac{\mathrm{SO}(6,4)}{\mathrm{SO}(6) \times \mathrm{SO}(4)} \tag{2.23}$$

$$\frac{\mathrm{SU}(4,3)}{\mathrm{SU}(4) \times \mathrm{SU}(3) \times \mathrm{U}(1)} \longrightarrow \frac{\mathrm{SU}(4,2)}{\mathrm{SU}(4) \times \mathrm{SU}(2) \times \mathrm{U}(1)} \tag{2.24}$$

with $(n_V, n_H) = (1, 6)$ and $(n_V, n_H) = (0, 4)$ respectively.

The first, together with its c-map $(n_V, n_H) = (5, 2)$

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,4)}{\mathrm{SO}(2) \times \mathrm{SO}(4)} \times \frac{\mathrm{SO}(4,2)}{\mathrm{SO}(4) \times \mathrm{SO}(2)} \tag{2.25}$$

corresponds to type IIB (type IIA) on T_6/Z_4.[5] The last is a truncation of the $(n_V, n_H) = (0, 10)$ case and its c-map is

$$\frac{\mathrm{SU}(3,1)}{\mathrm{SU}(3) \times \mathrm{U}(1)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2) \times \mathrm{U}(1)} \tag{2.26}$$

[5]Note that $\frac{\mathrm{SO}(4,2)}{\mathrm{SO}(4)\times SO(2)} \sim \frac{\mathrm{SU}(2,2)}{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}$.

JHEP03(2002)025

with $(n_V, n_H) = (3,1)$ (This is a truncation of the $(n_V, n_H) = (9,1)$ case.).

By the decomposition

$$\mathrm{SO}(6,6) \longrightarrow \mathrm{SO}(4,6-p) \times \mathrm{SO}(2,p) \tag{2.27}$$

we can obtain the additional cases:

$$\begin{aligned}
&(n_V = 2, n_H = 5) p = 1 \\
&(n_V = 4, n_H = 3) p = 3 \\
&(n_V = 3, n_H = 4) p = 2 \\
&(n_V = 6, n_H = 1) p = 5 \\
&(n_V = 7, n_H = 0) p = 6\,.
\end{aligned} \tag{2.28}$$

Their c-map do not give new models. We note that the case $p = 6$ is a truncation of the $(n_V, n_H) = (15,0)$ case and that the case $p = 5$ is not a subcase of the $(n_V, n_H) = (9,1)$ case because the corresponding quaternionic manifold is in this case $\frac{\mathrm{Usp}(2,2)}{\mathrm{Usp}(2)\times\mathrm{Usp}(2)}$ which is not the same of the $(n_V, n_H) = (9,1)$ case.

In conclusion we have found eleven "maximal" cases: the cases $(n_V, n_H) = (15,0), (6,1)$ which have no c-map counterpart, the case $(n_V, n_H) = (3,4)$ which is self conjugate under c-map and four pairs conjugate under c- map, namely:

$$\begin{aligned}
(n_V = 6, n_H = 2) &\overset{c-map}{\longleftrightarrow} (n_V = 1, n_H = 7) \\
(n_V = 5, n_H = 2) &\overset{c-map}{\longleftrightarrow} (n_V = 1, n_H = 6) \\
(n_V = 0, n_H = 10) &\overset{c-map}{\longleftrightarrow} (n_V = 9, n_H = 1) \\
(n_V = 4, n_H = 3) &\overset{c-map}{\longleftrightarrow} (n_V = 2, n_H = 5)\,.
\end{aligned} \tag{2.29}$$

Many of these cases can be retrieved from type II string theories compactified on $\mathbb{Z}_N$ orbifolds which preserve one left and one right supersymmetry [39, 41].

3. $D = 3$, $D = 5$ and $D = 6$ reduction of maximal supergravity to theories with eight supercharges

The same analysis can be carried out in $N = 2$ theories (eight supercharges) in $D = 3$, $D = 5$ (for the cases where the scalars span a symmetric space) and in $D = 6$.

In $D = 5$, $N = 8$ supergravity has a non-linear σ-model $E_{6(6)}/\,\mathrm{USp}(8)$ [33]. We consider only the $N = 8 \to N = 2$ case.

The 42 scalars, decomposed with respect to the $N = 2$ theory, consist of 14 scalars belonging to vector multiplets and $4 \times 7 = 28$ scalars belonging to quaternionic multiplets, giving $(n_V = 14, n_H = 0)$ and $(n_V = 0, n_H = 7)$ models which correspond to $\frac{\mathrm{SU}^*(6)}{\mathrm{USp}(6)} \subset \frac{E_{6(6)}}{\mathrm{USp}(8)}$ and $\frac{F_{4(4)}}{\mathrm{USp}(6)\times\mathrm{USp}(2)} \subset \frac{E_{6(6)}}{\mathrm{USp}(8)}$ [38]. For each model in $D = 4$ there is a parent in $D = 5$ (the above correspond to the $n_V \cdot n_H = 0$ cases).

If we now look to spaces with isometry groups $G_1 \times G_2 \subset E_{6(6)}$, where G_1, G_2 correspond to real special geometry and quaternionic geometry respectively, we find (see table 1):

$$G_1 \times G_2 = \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SU}(2,1) \tag{3.1}$$

JHEP03(2002)025

which give rise to

$$\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)} \times \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times\mathrm{U}(1)} \subset \frac{E_{6(6)}}{\mathrm{USp}(8)} \qquad (n_V = 8\,,\ n_H = 1) \tag{3.2}$$

and [40]

$$G_1 \times G_2 = \mathrm{SL}(3,\mathbb{R}) \times G_{2(2)} \tag{3.3}$$

giving

$$\frac{\mathrm{SL}(3,\mathbb{R})}{SO(3)} \times \frac{G_{2(2)}}{\mathrm{SO}(4)} \subset \frac{E_{6(6)}}{\mathrm{USp}(8)} \qquad (n_V = 5\,,\ n_H = 2)\,. \tag{3.4}$$

If we go through the $N = 4$ theory we also get the series of six cases

$$\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(1,p)}{\mathrm{SO}(p)}\times\frac{\mathrm{SO}(4,5-p)}{\mathrm{SO}(4)\times\mathrm{SO}(5-p)} \qquad (n_V = p+1\,,\ n_H = 5-p), \quad 0 \le p \le 5\,. \tag{3.5}$$

So we see that there are ten $D = 5$ cases with similar types of quaternionic manifold as in $D = 4$ (with the only exception of the $n_H = 10$ case.).

In $D = 6$ the $N = 8$ ((2,2) theory) σ-model is $\mathrm{SO}(5,5)/\,\mathrm{SO}(5)\times\mathrm{SO}(5)$. If we decompose the $(2,2)$ theory with respect to the $(1,0)$ theory we get 5 tensor multiplets and 5 hypermultiplets corresponding to

$$\frac{\mathrm{SO}(1,5)}{\mathrm{SO}(5)} \subset \frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5)\times\mathrm{SO}(5)}\,, \quad \frac{\mathrm{SO}(4,5)}{\mathrm{SO}(4)\times\mathrm{SO}(5)} \subset \frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5)\times\mathrm{SO}(5)}\,. \tag{3.6}$$

These are the $n_T \cdot n_H = 0$ cases.

Again we can now look at subgroups $G_1 \times G_2 \subset \mathrm{SO}(5,5)$ where $G_1 = \mathrm{SO}(1,n_T)$ and G_2 is the isometry group of a quaternionic manifold.

We find a series analogous to the $D = 5$ case (3.5), with

$$G_1 = \mathrm{SO}(1,p)\,, \quad G_2 = \mathrm{SO}(4,5-p) \qquad (n_T = p, n_V = 5-p) \tag{3.7}$$

corresponding to the manifolds

$$\frac{\mathrm{SO}(1,p)}{\mathrm{SO}(p)} \times \frac{\mathrm{SO}(4,5-p)}{\mathrm{SO}(4)\times\mathrm{SO}(5-p)} \qquad (n_T = p\,,\ n_H = 5-p), \quad 0 \le p \le 5 \tag{3.8}$$

which contains also the above mentioned $n_T \cdot n_H = 0$ cases (3.6).

The reduction of $N = 8 \to N = 2$ supergravity studied in $D = 6, 5$ and 4 finds a further simplification if we look for theories with eight supercharges in $D = 3$, where the R-symmetry is $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$.

In fact, if we compactify type II on a Calabi-Yau threefold times S_1, down to $D = 3$, then type IIA and IIB become the same theory with $1 \Leftrightarrow 2$. The $N = 4$ σ-model is a product of two quaternionic geometries, where $n_{H_1} = h_{1,1} + 1$, $n_{H_2} = h_{2,1} + 1$, the extra quaternion coming from the graviton and graviphoton degrees of freedom.

More generically, suppose we have a theory which at $D = 4$ has a σ-model $\mathcal{M}^{SK}(n_V) \times \mathcal{M}^{Q}(n_H)$, then its dimensional reduction to $D = 3$ will give rise to a $N = 4$ $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ theory with σ-model $\mathcal{M}^{Q_1}(n_{H_1} = n_V + 1) \times \mathcal{M}^{Q_2}(n_{H_2} = n_H)$, where $\mathcal{M}^{Q_1}$ is the dual quaternionic manifold of $\mathcal{M}^{SK}(n_V)$.

JHEP03(2002)025

Using the previous recipe, if we look to the $D=4$, $N=2$ theories of section 2 obtained from $N=8$, we can predict $N=4$ theories at $D=3$ which are embedded in the $\frac{E_{8(8)}}{SO(16)}$ σ-model of $D=3$, $N=16$ maximal supergravity.

From $(n_V=15, n_H=0)$ and $(n_V=0, n_H=10)$ we respectively obtain:

$$\begin{aligned}
&(n_{H_1}=16, n_{H_2}=0)\,, \frac{E_{7(-5)}}{SO(12)\times \mathrm{SU}(2)} \subset \frac{E_{8(8)}}{\mathrm{SO}(16)} \\
&(n_{H_1}=1, n_{H_2}=10)\,, \frac{\mathrm{SU}(2,1)}{\mathrm{SU}(2)\times \mathrm{U}(1)} \times \frac{E_{6(2)}}{\mathrm{SU}(6)\times \mathrm{SU}(2)} \subset \frac{E_{8(8)}}{\mathrm{SO}(16)} \\
&(n_{H_1}=2, n_{H_2}=7)\,, \frac{G_{2(2)}}{\mathrm{SO}(4)} \times \frac{F_{4(4)}}{\mathrm{USp}(6)\times \mathrm{USp}(2)} \subset \frac{E_{8(8)}}{\mathrm{SO}(16)}\,.
\end{aligned} \tag{3.9}$$

Also, by using the embedding [42] $E_{8(8)} \supset \mathrm{SO}(8,8)$ we have the further possibility:

$$\frac{E_{8(8)}}{\mathrm{SO}(16)} \supset \frac{\mathrm{SO}(8,8)}{\mathrm{SO}(8)\times \mathrm{SO}(8)} \supset \frac{\mathrm{SO}(4,k)\times \mathrm{SO}(4,8-k)}{\mathrm{SO}(4)\times \mathrm{SO}(k)\times \mathrm{SO}(4)\times \mathrm{SO}(8-k)} \tag{3.10}$$

with $n_{H_1}=k$, $n_{H_2}=8-k$ ($k=0$ is a subcase of the $(n_{H_1}=16, n_{H_2}=0)$ case, since $\mathrm{SO}(4,8)\times \mathrm{SU}(2) \subset E_{7(-5)}$.). They are all dimensional reductions of the cases previously studied at $D=4$.

For the decomposition of the isometry group of maximal $D=3$ supergravity to maximal subgroups, see table 1.

4. Geometrical interpretation

It is interesting to analyze the results of the previous section in geometrical terms, that is to explore the consistency of the reduction of the $N=8$ σ-model $E_{7(7)}/\,\mathrm{SU}(8)$ to the appropriate submanifolds for different values of N'. A consistent truncation of a manifold of dimension n to a submanifold of dimension $n-k$ can be obtained by considering a set of k 1-forms ϕ^i, $i=1,\ldots,k$, which vanish on the submanifold and such that they are in involution, that is:

$$d\phi^i = \theta^i_j \wedge \phi^j\,, \tag{4.1}$$

where θ^i_j are suitable 1-forms on the manifold.

To apply this result, known as Frobenius theorem, to our problem we consider the coset representative U of $E_{7(7)}/\,\mathrm{SU}(8)$ in the **56** fundamental representation of $E_{7(7)}$ and the corresponding left invariant 1-form:[6]

$$\Gamma \equiv U^{-1}dU = \begin{pmatrix} \Omega & \overline{P} \\ P & \overline{\Omega} \end{pmatrix} \tag{4.2}$$

satisfying the Cartan-Maurer equation

$$d\Gamma + \Gamma\wedge\Gamma = 0\,. \tag{4.3}$$

[6]We use notations as in ref. [43].

JHEP03(2002)025

Some Maximal subgroups of $E_{8(8)}$:
$\mathrm{SO}(16)$
$\mathrm{SO}(8,8)$
$E_{7(-5)} \times \mathrm{SU}(2)$
$E_{6(2)} \times \mathrm{SU}(2,1)$
$G_{2(2)} \times F_{4(4)}$
Some Maximal subgroups of $E_{7(7)}$:
$\mathrm{SU}(8)$
$\mathrm{SU}(4,4)$
$E_{6(2)} \times \mathrm{SO}(2)$
$\mathrm{SO}^*(12) \times \mathrm{SU}(2)$
$\mathrm{SO}(6,6) \times \mathrm{SU}(1,1)$
$\mathrm{SU}(3,3) \times \mathrm{SU}(2,1)$
$\mathrm{SU}(1,1) \times F_{4(4)}$
$\mathrm{Sp}(6,\mathbb{R}) \times G_{2(2)}$
Some Maximal subgroups of $E_{6(6)}$:
$\mathrm{USp}(8)$
$F_{4(4)}$
$\mathrm{SU}^*(6) \times \mathrm{SU}(2)$
$\mathrm{SO}(5,5) \times \mathrm{SO}(1,1)$
$\mathrm{SL}(3,\mathbb{R}) \times G_{2(2)}$
$\mathrm{SL}(3,\mathbb{C}) \times \mathrm{SU}(2,1)$

Table 1: Decomposition of "duality" groups of maximal $D = 3,4,5$ supergravities with respect to maximal subgroups relevant for supergravity reduction [42, 40].

Here the 28×28 subblocks Ω and P embed the SU(8) connection and the vielbein of $E_{7(7)}/\,\mathrm{SU}(8)$. Introducing indices $A, B = 1, \ldots, 8$ we have explicitly:

$$\Omega \equiv 2\omega^{[A}{}_{[C}\delta^{B]}_{D]}\,; \qquad\qquad P \equiv P_{ABCD}\,, \tag{4.4}$$

where $\omega^A{}_B$ is the SU(8) connection and P_{ABCD} is the vielbein of $E_{7(7)}/\,\mathrm{SU}(8)$, antisymmetric in its four indices and satisfying the reality condition:

$$P_{ABCD} = \frac{1}{24}\epsilon_{ABCDPQRS}\overline{P}^{PQRS}\,. \tag{4.5}$$

From the Cartan-Maurer equations one easily finds the two structure equations:

$$R^A{}_B \equiv d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B = -\frac{1}{3}\overline{P}^{ALMN} \wedge P_{BLMN} \tag{4.6}$$

$$\nabla\overline{P}^{ABCD} \equiv d\overline{P}^{ABCD} - 4\omega^{[A}{}_L\overline{P}^{BCD]L} = 0\,. \tag{4.7}$$

Equation (4.6) gives the SU(8) Lie algebra valued curvature $R^A{}_B$ in terms of the vielbein of the symmetric coset $E_{7(7)}/\,\mathrm{SU}(8)$ and equation (4.7) expresses the fact that the same

JHEP03(2002)025

manifold is torsionless. Note that, since the coset is symmetric, the Lie algebra connection $\omega^A{}_B$ is simply related via a structure constant to the riemannian spin connection.

Let us now consider how the vielbein P_{ABCD} decomposes under the holonomy reduction $\mathrm{SU}(8) \longrightarrow \mathrm{SU}(N')\times U(1)\times\mathrm{SU}(8-N')$. We call $a,b,c,\ldots = 1,\ldots,N'$ the indices of $\mathrm{SU}(N')$ and $i,j,k\ldots = 1,\ldots,8-N'$ the indices of $\mathrm{SU}(8-N')$. Then the holonomy reduction gives the following fragments:

$$P_{ABCD} \longrightarrow P_{abcd} \oplus P_{abci} \oplus P_{abij} \oplus P_{aijk} \oplus P_{ijkl}\,, \tag{4.8}$$

where actually some of the fragments can be zero if the number of antisymmetric indices of $\mathrm{SU}(N')$ or $\mathrm{SU}(8-N')$ exceeds N' or $8-N'$, respectively. Now we observe that P_{abcd} satisfies eq. (4.7) which gives for this particular component:

$$d\overline{P}^{abcd} - 4\omega^{[a}{}_{\ell}\overline{P}^{bcd]\ell} - 4\omega^{[a}{}_{i}\overline{P}^{bcd]i} = 0\,. \tag{4.9}$$

We see that, in order that eq. (4.9) describe a torsionless submanifold with $\mathrm{SU}(N')\times\mathrm{U}(1)$ holonomy, we must set $\omega^a{}_i = 0$ and since

$$R^a_{\ i} \equiv d\omega^a{}_i + \omega^a{}_c\wedge\omega^c{}_i + \omega^a{}_j\wedge\omega^j{}_i = -\frac{1}{3}\overline{P}^{aLMN}\wedge P_{iLMN} \tag{4.10}$$

we must also impose that, on the submanifold whose vielbeins are P^{abcd}, the curvature with mixed indices is zero, namely $R^a_{\ i} = -\frac{1}{3}\overline{P}^{aLMN}\wedge P_{iLMN} = 0$. Using the decomposition (4.8), eq. (4.10) can be rewritten as follows

$$\begin{aligned} d\omega^a{}_i &= -\omega^a{}_c\wedge\omega^c{}_i - \omega^a{}_j\wedge\omega^j{}_i - \frac{1}{3}\overline{P}^{abcd}\wedge P_{ibcd} - \\ &\quad -\frac{1}{3}\overline{P}^{abcj}\wedge P_{ibcj} - \frac{1}{3}\overline{P}^{abjk}\wedge P_{ibjk} - \frac{1}{3}\overline{P}^{ajkl}\wedge P_{ijkl}\,. \end{aligned} \tag{4.11}$$

On the basis of the Frobenius theorem, each term on the r.h.s. of (4.11) must be in involution with $\omega^a{}_i$; this is satisfied for the terms bilinear in the ω-connections, but not for those involving the vielbein. In order to obtain involution, we must also set to zero some of the vielbein 1-forms and verify that also these are actually in involution. Let us see how we can achieve this result in the various cases.

When $N'=6$, $P_{ijkl} = P_{ibjk} \equiv 0$ because we have 4-fold or threefold antisymmetrization of the $\mathrm{SU}(2)$ indices. Therefore it is sufficient to set

$$P_{ibcd} \equiv \overline{P}^{ibcd} = 0 \tag{4.12}$$

on the submanifold in order to obtain involution, since in this case eq. (4.11) reduces to

$$d\omega^a{}_i = -\omega^a{}_c\wedge\omega^c{}_i - \omega^a{}_j\wedge\omega^j{}_i \longrightarrow R^a_{\ i} = 0\,. \tag{4.13}$$

We still have to verify that also the vanishing 1-forms P_{ibcd} are in involution with themselves and with $\omega^a{}_i$. Indeed, from eq. (4.7), we find:

$$d\overline{P}^{abci} = 3\omega^{[a}{}_{d}\overline{P}^{bc]di} + 3\omega^{[a}{}_{j}\overline{P}^{bc]ji} + \omega^i{}_d\overline{P}^{abcd} + \omega^i{}_j\overline{P}^{abcj} \tag{4.14}$$

JHEP03(2002)025

and we see that every term in the r.h.s. contains either $\overline{P}^{abcj}$ or $\omega^a_{\ i}$ so that we get involution.

We note that condition (4.12) is equivalent to impose that the $\mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{SU}(2)$ representation $(\mathbf{20}, \mathbf{0}, \mathbf{2})$ must be absent in the reduction of the scalar vielbein, and this implies that all the 40 scalars of the $N' = 6$ spin 3/2 multiplets must be frozen according to our analysis in the previous section. In conclusion, setting $P^{abcj} = 0$ and $\omega^a_{\ i} = 0$, we define a consistent truncation of the $N = 8$ theory down to a $N' = 6$ theory since the above conditions define a submanifold of holonomy $\mathrm{SU}(6) \times \mathrm{U}(1) \times \mathrm{SU}(2)$ whose curvature is easily seen to be given by

$$R^a_{\ b} = -\frac{1}{3} \overline{P}^{almn} \wedge P_{blmn} \,. \tag{4.15}$$

The corresponding manifold has dimension 30 and of course coincides with $\mathrm{SO}^*(12)/\,\mathrm{U}(6)$.

The cases $N' = 5$ and $N' = 4$ can be treated in exactly the same way. For $N' = 5$, eq. (4.11) does not contain P_{ijkl} and in order to get involution we have to set

$$P_{abci} = P_{abij} = 0 \tag{4.16}$$

which corresponds to delete, in the holonomy reduction (2.2), the representations $(\mathbf{10}, \mathbf{1}, \mathbf{3})$ and $(\overline{\mathbf{10}}, -\mathbf{1}, \overline{\mathbf{3}})$ for the vielbein (because of the reality condition(4.5)). According to the discussion of the previous section, this is equivalent to freeze the 60 scalars of the spin 3/2 multiplets. Using again eq. (4.7), we can immediately verify that P_{abci}, P_{abij} and $\omega^a_{\ i}$ are indeed in involution so that the reduction to the submanifold $\mathrm{SU}(1,5)/\,\mathrm{U}(5)$ is indeed consistent.

For $N' = 4$, eq. (4.11) contains all the terms bilinear in the vielbeins. However it is sufficient to set

$$P_{abci} = P_{aijk} (\approx \overline{P}^{abci}) = 0 \tag{4.17}$$

to achieve the vanishing of the r.h.s. of (4.11) on the submanifold. This corresponds to delete, in the holonomy reduction (2.5), the representations $(\mathbf{4}, \mathbf{1}, \mathbf{4})$ and $(\overline{\mathbf{4}}, -\mathbf{1}, \overline{\mathbf{4}})$ in the decomposition of the scalar vielbein, that is to freeze the 32 scalars appearing in the $N' = 4$ spin 3/2 multiplets. Again the structure eq. (4.7) can be used to show that P_{abci}, P_{aijk} and $\omega^a_{\ i}$ are in involution so that we get a consistent reduction to the $N' = 4$ submanifold $\mathrm{SU}(1,1)/\,\mathrm{U}(1) \times \mathrm{SO}(6,6)/[\mathrm{SU}(4) \times \mathrm{SU}(4)]$.

The reduction to the submanifold of the $N' = 3$ theory requires a little more labor. In this case equation (4.11) does not contain the term $\overline{P}^{abcd} \wedge P_{ibcd}$ and if we set

$$P^{abci} = P_{ijkl} = 0 \tag{4.18}$$

then

$$R^a_{\ i} = -\frac{1}{3} \overline{P}^{abjk} \wedge P_{ibjk} \neq 0 \,. \tag{4.19}$$

We could of course set also $\overline{P}^{abjk} = 0$, but then we would be left with a theory without scalars, that is pure $N' = 3$ supergravity theory.

In order to obtain a matter coupled $N' = 3$ theory, we further reduce the submanifold holonomy:

$$\mathrm{SU}(8) \longrightarrow \mathrm{SU}(3) \times U(1) \times \mathrm{SU}(5) \longrightarrow \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SU}(4) \,. \tag{4.20}$$

JHEP03(2002)025

To see that in this case we obtain a consistent submanifold, we split the SU(5) indices $i, j, \ldots = 1, \ldots, 5$ into SU(4) indices $\alpha, \beta, \ldots = 1, \ldots, 4$ and the index 5. Then we have:

$$
\begin{aligned}
& R^a_{\ i} \longrightarrow R^a_{\ \alpha};\ R^a_{\ 5} \\
R^a_{\ \alpha} &= -\frac{1}{3}\overline{P}^{ab\beta\gamma} \wedge P_{\alpha b\beta\gamma} - \frac{2}{3}\overline{P}^{ab\beta 5} \wedge P_{\alpha b\beta 5} \\
R^a_{\ 5} &= -\frac{1}{3}\overline{P}^{ab\beta\gamma} \wedge P_{5b\beta\gamma}\,.
\end{aligned}
\tag{4.21}
$$

The vielbeins $P_{ab\alpha\beta}$ and $P_{5\beta ab}$ are in the representations $(\overline{\mathbf{3}}, \mathbf{6})$ and $(\overline{\mathbf{3}}, \mathbf{4})$ of $\mathrm{SU}(3) \times \mathrm{SU}(4)$, respectively. Hence if we delete the representation $(\overline{\mathbf{3}}, \mathbf{6})$, that is if we set

$$
P_{ab\alpha\beta} = P_{5b\beta\gamma} = 0 \tag{4.22}
$$

we get $R^a_{\ \alpha} = R^a_{\ 5} = 0$. On the light of the discussion given in the previous section for the same case, this corresponds to select, as different from zero on the submanifold, only the vielbeins with indices in the $(\overline{\mathbf{3}}, \mathbf{4})$ rep. of the holonomy group $\mathrm{U}(3) \times \mathrm{SU}(4)$. We obtain in this case a consistent reduction to the submanifold spanned by the vielbeins $P_{ab\gamma 5}$ since it can be easily verified that $P_{abc\alpha}, P_{abc5}, P_{\alpha\beta\gamma\delta}, P_{\alpha\beta\gamma 5}, \omega^a_{\ \alpha}, \omega^a_{\ 5}$ are all in involution among themselves.

Finally, in the $N' = 2$ case, in order to have involution for $\omega^a_{\ i} = 0$, we must have

$$
R^a_{\ i} = -\frac{1}{3}\overline{P}^{abjk} \wedge P_{ibjk} - \frac{1}{3}\overline{P}^{ajkl} \wedge P_{ijkl} = 0 \tag{4.23}
$$

on the manifold.

If we take $\overline{P}^{abjk}$ (and its complex conjugate $P_{ijk\ell}$) or $\overline{P}^{ajkl}$ vanishing on the submanifold this corresponds to delete the complex representation $(\overline{\mathbf{1}}, -\mathbf{1}, \mathbf{15})$ or the real representation $(\mathbf{2}, \mathbf{0}, \mathbf{20})$ of the holonomy group $\mathrm{SU}(2) \times U(1) \times \mathrm{SU}(6)$. We may check immediately that in both cases the vanishing vielbein are indeed in involution with $\omega^a_{\ i}$ and with themselves. Indeed:

$$
\begin{aligned}
d\overline{P}^{abci} &= 3\omega^{[a}_{\ \ d}\overline{P}^{bc]di} + 3\omega^{[a}_{\ \ j}\overline{P}^{bc]ji} + \omega^i_{\ d}\overline{P}^{abcd} + \omega^i_{\ j}\overline{P}^{abcj} \\
d\overline{P}^{ajk\ell} &= \omega^a_{\ b}\overline{P}^{bjk\ell} + \omega^a_{\ i}\overline{P}^{ijk\ell} + 3\omega^{[j}_{\ \ b}\overline{P}^{k\ell]ab} + 3\omega^{[j}_{\ \ i}\overline{P}^{k\ell]ia}
\end{aligned}
\tag{4.24}
$$

and we see that in both cases the involution condition is satisfied. Therefore we have found a consistent reduction to the submanifolds $\mathrm{SO}^*(12)/\,\mathrm{U}(6)$ and $E_{6(2)}/\,\mathrm{SU}(6) \times \mathrm{SU}(2)$ which are special-Kähler and quaternionic manifolds respectively of maximal holonomy.

The other cases treated group theoretically in the previous section can be handled in an analogous way, provided we reduce the holonomy of the resulting submanifold in a suitable way. We just give an example.

Consider the manifold given in eq. (2.19), corresponding to $(n_V, n_H) = (9, 1)$. We decompose the representation $\mathbf{6}$ of SU(6) into the representation $(\mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{3})$ of $\mathrm{SU}(3) \times \mathrm{SU}(3)$. Correspondingly, the index i in the $\mathbf{6}$ of SU(6) is decomposed:

$$
i \to \alpha, \dot\alpha\,, \qquad (\alpha, \dot\alpha = 1, 2, 3)\,, \tag{4.25}
$$

JHEP03(2002)025

where α and $\dot\alpha$ run on the fundamental rep of the two SU(3) groups. Then we have:

$$R^a{}_i \to R^a{}_\alpha \,,\, R^a{}_{\dot\alpha} \tag{4.26}$$

and we find:

$$\begin{aligned} R^a{}_\alpha = & \overset{(1,\overline{3},1)}{\overline{P}}{}^{ab\beta\gamma} \wedge \overset{(2,1,1)}{P}_{\alpha b\beta\gamma} + \overset{(1,3,3)}{\overline{P}}{}^{ab\beta\dot\gamma} \wedge \overset{(2,\overline{3},3)}{P}_{\alpha b\beta\dot\gamma} + \overset{(1,1,\overline{3})}{\overline{P}}{}^{ab\dot\beta\dot\gamma} \wedge \overset{(2,3,\overline{3})}{P}_{\alpha b\dot\beta\dot\gamma} + \\ & + \overset{(2,\overline{3},3)}{\overline{P}}{}^{a\beta\gamma\dot\delta} \wedge \overset{(1,1,3)}{P}_{\alpha\beta\gamma\dot\delta} + \overset{(2,3,\overline{3})}{\overline{P}}{}^{a\beta\dot\gamma\dot\delta} \wedge \overset{(1,\overline{3},\overline{3})}{P}_{\alpha\beta\dot\gamma\dot\delta} + \overset{(2,1,1)}{\overline{P}}{}^{a\dot\beta\dot\gamma\dot\delta} \wedge \overset{(1,3,1)}{P}_{\alpha\dot\beta\dot\gamma\dot\delta} \,, \end{aligned} \tag{4.27}$$

where we have set on the top of each vielbein the rep of $\mathrm{SU}(2) \times \mathrm{SU}(3) \times \mathrm{SU}(3)$ to which it belongs. We see that deleting the vielbein in the reps $(\mathbf{1}, \overline{\mathbf{3}}, \mathbf{1})$, $(\mathbf{2}, \overline{\mathbf{3}}, \mathbf{3})$ and $(\mathbf{1}, \mathbf{1}, \overline{\mathbf{3}})$ (and their complex conjugates) we get $R^a{}_\alpha = 0$ so that involution is satisfied. An analogous computation can be done, with the same conclusions, for $R^a{}_{\dot\alpha}$. Note that the vielbein which survive, $P_{a\alpha\beta\gamma}$ and $P_{ab\beta\dot\gamma}$, in the representations $(\mathbf{2}, \mathbf{1}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{3}, \mathbf{3})$ respectively, do in fact describe the vielbein system of the given manifold.

The involution of the deleted vielbein is also easily proved. Indeed:

$$d\overline{P}^{ab\beta\gamma} = 2\omega^{[a}{}_c \wedge \overline{P}^{b]c\beta\gamma} + 2\omega^{[a}{}_\alpha \wedge \overline{P}^{b]\alpha\beta\gamma} + 2\omega^{[a}{}_{\dot\alpha} \wedge \overline{P}^{b]\dot\alpha\beta\gamma} - 2\omega^{[\beta}{}_\delta \wedge \overline{P}^{\gamma]\delta ab} - 2\omega^{[\beta}{}_{\dot\delta} \wedge \overline{P}^{\gamma]\dot\delta ab} \tag{4.28}$$

and we see that each term contains at least a 1-form which is zero on the submanifold.

It is a simple exercise to verify that one can actually further reduce the holonomy to all the holonomy subgroups of the various cases treated in section 2 and find consistent reduction to the corresponding special-Kähler and quaternionic symmetric coset submanifolds.

5. Consistency constraints from supersymmetry

In the previous sections we have analyzed the effects of truncating out some of the supercharges in the supergravity theories. In particular, in section 3 we have considered the effects of the reduction of the holonomy group for the various supermultiplets at the linearized level, while in section 4 we have studied the consequences of such a reduction on the scalar sectors.

We still have to analyze if the consistency found at the level of σ-model in the geometrical analysis can be extended to the full supersymmetric level.

For this purpose, we analyze the supersymmetry transformation laws of $N = 8$ supergravity, when the R-symmetry gets reduced from SU(8) to $\mathrm{SU}(N') \times \mathrm{U}(1)$. They are, neglecting three fermions terms:

$$\begin{aligned} \delta V^a_\mu &= -\mathrm{i}\overline{\psi}^A_\mu \gamma^a \epsilon_A + \mathrm{h.c.} \\ \delta\psi_{A\mu} &= \nabla_\mu \epsilon_A + T^-_{AB|\nu\rho} \gamma_\mu{}^\nu \gamma^\rho \epsilon^B \\ \delta\chi_{ABC} &= P_{ABCD,\alpha} \partial_\mu \phi^\alpha \gamma^\mu \epsilon^D + T^-_{[AB|\mu\nu} \gamma^{\mu\nu} \epsilon_{C]} \\ \delta A^{\Lambda\Sigma}_\mu &= f^{\Lambda\Sigma}_{AB} \left(\overline{\psi}^A_\mu \epsilon^B + \overline{\chi}^{ABC} \gamma_\mu \epsilon_C \right) + \mathrm{h.c.} \\ \delta\phi^\alpha &= \overline{P}^{ABCD,\alpha} \overline{\chi}_{ABC} \epsilon_D + \mathrm{h.c.} \end{aligned} \tag{5.1}$$

JHEP03(2002)025

(the SU(8) indices $A, \ldots$ run from 1 to 8). We use the same notation as in reference [43]: we call U the coset representative of $E_{7(7)}/\,\mathrm{SU}(8)$ parametrized as follows:

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} f + \mathrm{i}h & \overline{f} + \mathrm{i}\overline{h} \\ f - \mathrm{i}h & \overline{f} - \mathrm{i}\overline{h} \end{pmatrix}, \tag{5.2}$$

where $f^{\Lambda\Sigma}_{AB}$ and $h_{\Lambda\Sigma AB}$ $(\Lambda, \Sigma, \ldots = 1, \ldots, 8)$ are labelled by couples of antisymmetric indices $\Lambda\Sigma$ andAB with $\Lambda, \Sigma = 1 \ldots, 8$ and $A, B = 1 \ldots, 8$. Therefore they describe 28×28 sub-blocks of the 56×56 symplectic matrix (coinciding with the fundamental **56** representation of $E_{7(7)}$). Note that U transforms on the left as the **56** representation of $E_{7(7)}$ and on the right as the $\mathbf{28} \oplus \overline{\mathbf{28}}$ of SU(8) .

In terms of f and h, the 2-form T_{AB} is given by:

$$T_{AB} = -\frac{\mathrm{i}}{2} \left(\overline{f}^{-1}\right)_{AB\Lambda\Sigma} F^{\Lambda\Sigma} = \frac{1}{2} \left(h_{\Lambda\Sigma AB} F^{\Lambda\Sigma} - f^{\Lambda\Sigma}_{AB} \mathcal{G}_{\Lambda\Sigma}\right), \tag{5.3}$$

where $\mathcal{G}_{\Lambda\Sigma}$ is the magnetic counterpart of the field-strength $F^{\Lambda\Sigma}$. The spinor fields $\psi_{A\mu}$ and χ_{ABC} are the $N = 8$ left-handed gravitinos and dilatinos respectively. Finally, the covariant derivative acting on the spinors is defined as follows:

$$\nabla \epsilon_A = \mathcal{D}\epsilon_A + \omega_A{}^B \epsilon_B \,, \tag{5.4}$$

where $\omega_A{}^B$ is the SU(8) connection and $\mathcal{D}_\mu$ denotes the Lorentz covariant derivative.

Let us first analyze the gravitino decomposition. We want to reduce the theory to an $N' \leq 8$ one. Therefore, to reduce the R-symmetry $\mathrm{SU}(8) \to \mathrm{SU}(N') \times \mathrm{U}(1)$, we decompose the holonomy indices $A, \ldots \Rightarrow (a, i)$ with $a = 1, \ldots, N'$ and $i = 1, \ldots, 8 - N'$. We then have to truncate out (to set to zero) the $8 - N'$ gravitinos $\psi_{i\mu}$ and the corresponding supersymmetry parameters ϵ_i. We get:

$$\begin{aligned} \delta\psi_{a\mu} &= \mathcal{D}_\mu \epsilon_a + \omega_a{}^b \epsilon_b + T^-_{ab|\nu\rho} \gamma_\mu{}^\nu \gamma^\rho \epsilon^b \\ \delta\psi_{i\mu} &= \omega_i{}^a \epsilon_a + T^-_{ia|\nu\rho} \gamma_\mu{}^\nu \gamma^\rho \epsilon^a \equiv 0 \,. \end{aligned} \tag{5.5}$$

The second equation, consistency condition for the truncation, implies

$$\omega_i{}^a = 0 \,, \qquad T^-_{ia} = 0 \,. \tag{5.6}$$

The first condition in (5.6) confirms the restriction of the scalar σ-models found in the previous section from the geometrical analysis, while the second one kills the vector super-partners of the erased gravitinos at the full interaction level.

Then what is left, eq. (5.5), is the correct transformation law for the survived gravitini, provided $T_{ab} = -\frac{\mathrm{i}}{2}(\overline{f}^{\,-1})_{ab\Lambda\Sigma} F^{\Lambda\Sigma}$ (and $T_{ij} = -\frac{\mathrm{i}}{2}(\overline{f}^{\,-1})_{ij\Lambda\Sigma} F^{\Lambda\Sigma}$ for $N' = 6$) describe the correct expression for the (dressed) graviphotons in the reduced theory, $T_{ab} = -\frac{\mathrm{i}}{2}(\overline{f}^{\,-1})_{ab\boldsymbol{\Lambda}} F^{\boldsymbol{\Lambda}}$, with $\boldsymbol{\Lambda}$ running on the appropriate representation of the U duality group of the reduced theory.[7]

[7] With abuse of language, we call U duality group the continuous group whose restriction to the integers is the U duality group of the theory.

JHEP03(2002)025

$G = E_{7(7)}$	$G_1 \times G_2$	$G \to G_1 \times G_2$
$N = 6$ (# vect. $= 16$)	$\mathrm{SO}^*(12) \times \mathrm{SU}(2)$	$56 \longrightarrow (32,1) + (12,2)$
$N = 5$ (# vect. $= 10$)	$\mathrm{SU}(5,1) \times \mathrm{SU}(3)$	$56 \to (10,1) + (\overline{10},1) + (6,3) + (\overline{6},\overline{3})$
$N = 4$ (# vect. $= 12$)	$\mathrm{SO}(6,6) \times \mathrm{SU}(1,1)$	$56 \longrightarrow (12,2) + (32,1)$
$N = 3$ (# vect. $= 4$)	$\mathrm{SU}(3,4) \times \mathrm{U}(1)$	$56 \longrightarrow 21 + \overline{21} + 7 + \overline{7}$
$N = 2$ (# vect. $= n_V + 1$)		
$n_V = 0$	$E_{6(2)} \times \mathrm{U}(1)$	$56 \to 1 + 1' + 27 + \overline{27}$
$n_V = 15$	$\mathrm{SO}^*(12) \times \mathrm{SU}(2)$	$56 \to (32,1) + (12,2)$
$n_V = 9$	$\mathrm{SU}(3,3) \times \mathrm{SU}(2,1)$	$56 \to (20,1) + (6,3) + (\overline{6},\overline{3})$
$n_V = 6$	$\mathrm{Sp}(6,\mathbb{R}) \times G_{2(2)}$	$56 \to (1,14) + (6,7)$
$n_V = 2$	$\mathrm{SU}(1,1) \times F_{4(4)}$	$56 \to (4,1) + (2,26)$

Table 2: Duality reduction in $D = 4$.

To this aim, let us first recall that, in all N-extended theories, the electric and magnetic field-strengths transform in a representation of the U duality group whose dimension is the same as the fundamental representation of the embedding symplectic group $\mathrm{Sp}(2n_v)$ [44] (n_v is the total number of vectors). Let us consider separately the cases $N' = 5, 6$, where all the vectors are graviphotons, from the $N' \leq 4$ cases, where matter vectors are present.

In the former cases, note that $E_{7(7)}$ (the isometry group of $N = 8$ theory) contains, as maximal subgroups: $\mathrm{SO}^*(12) \times \mathrm{SU}(2)$ and $\mathrm{SU}(5,1) \times \mathrm{SU}(3)$. The duality groups for the $N' = 6, 5$ are $\mathrm{SO}^*(12)$ and $\mathrm{SU}(5,1)$ respectively. The rep **56**, in which the $N = 8$ vectors field strengths and their duals lie, decomposes respectively as follows (see also table 2):

$$E_{7(7)} \to \overset{*}{\mathrm{SO}}(12) \times \mathrm{SU}(2) \mathbf{56} \to (\mathbf{32}, \mathbf{1}) + (\mathbf{12}, \mathbf{2}) \tag{5.7}$$

$$E_{7(7)} \to \mathrm{SU}(5,1) \times \mathrm{SU}(3) \mathbf{56} \to (\mathbf{20}, \mathbf{1}) + (\mathbf{6}, \mathbf{3}) + (\overline{\mathbf{6}}, \overline{\mathbf{3}})\,. \tag{5.8}$$

We note that in each case only a subset of the 56 field-strengths is transformed only with respect to the (reduced theory) duality group, while it is a singlet of the $\mathrm{SU}(8 - N')$ commuting group, and this immediately identifies the electric and magnetic field strengths which remain in the gravitational multiplet after truncation. (Indeed this exactly reproduces the counting at the linearized level, since we expect to have, in the gravitational multiplet of the $N' = 6$ (respectively $N' = 5$) theory, 16 (respectively 10) electric field strengths parametrized by T_{ab}, T_{ij} (respectively by T_{ab}).

Therefore, in performing the truncation, we also have to decompose the representations of the $N = 8$ U duality group with respect to its maximal subgroups as in (5.7), (5.8), and to keep only the irrepses, in the decomposition, which are singlets under the commuting group, as shown in table 2.

Note that this prescription automatically guarantees the consistency of the truncation, since the objects to be truncated out (in particular the $(12, 2)$ (respectively $(\mathbf{6}, \mathbf{3}) + (\overline{\mathbf{6}}, \overline{\mathbf{3}})$) field strengths given by T_{ai} and their magnetic duals), being in a non trivial representation of the commuting group $\mathrm{SU}(8 - N')$, can never mix with those which have been kept, which are instead singlets.

JHEP03(2002)025

Let us now consider the matter coupled theories, and in particular $N' = 4$ (the $N' = 2$ case is similar). Here the argument is reversed with respect to the higher N' theories, but with analogous conclusions. Indeed, the U duality group for the $N' = 4$ theory is $\mathrm{SU}(1,1) \times \mathrm{SO}(6,n)$, and, for $n = 6$, it is indeed a maximal subgroup of the $N = 8$ U duality group, (no commuting subgroup). Note that the U-duality group is now factorized into the S-duality group $\mathrm{SU}(1,1)$, which mixes electric with magnetic field strengths, and the electric T-duality group $\mathrm{SO}(6,6)$. We have, for the decomposition of the **56** of $E_{7(7)} \to \mathrm{SU}(1,1) \times \mathrm{SO}(6,6)$:

$$\mathbf{56} \to (\mathbf{2},\mathbf{12}) + (\mathbf{1},\mathbf{32})\,. \tag{5.9}$$

In this case it is the $(\mathbf{2},\mathbf{12})$ field strengths (given by the six graviphotons T_{ab} and the six matter vectors T_{ij}, together with their magnetic counterpart) which have to be retained, since they have the appropriate transformation property under the U duality group, while the extra 32 field-strengths (given by T_{ai} and its magnetic dual), which are spinors under $\mathrm{SO}(6,6)$, have to be truncated out and do indeed belong to the extra gravitini multiplets. A similar argument as given previously still works for the consistency; indeed the field-strengths in the $(\mathbf{1},\mathbf{32})$, spinors under $\mathrm{SO}(6,6)$, can be set to zero consistently since they cannot mix with the other field-strengths which are not in the spinor representation of $\mathrm{SO}^*(12)$. As far as the transformation laws for the vectors, scalars and spin one half fields are concerned, one sees that the decomposition confirms the results of the analysis at the linearized level given in section 3, as summarized in table 3.

For the case $N = 8 \to N = 2$, we see from table 2 that the vectors belonging to the six spin 3/2 multiplets and to those vector multiplets which are truncated out are tied together by an irrep. of $G_1 \times G_2$. This means that to delete only the spin 3/2 multiplets would be inconsistent.

The same analysis applies to theories in higher dimensions and, for the $D = 5$ case, the duality reduction, for some interseting cases, is given in table 4.

6. $N = 2 \longrightarrow N = 1$ reduction

This section is devoted to a thorough analysis of the consistent truncation of $N = 2$ supergravity down to $N = 1$ in four dimensions. The $N = 2 \longrightarrow N = 1$ reduction of the supersymmetry transformation laws presents different features in the vector multiplet and in the hypermultiplet sectors. The vector multiplet case is simpler since the special geometry is already a Kähler-Hodge geometry while for hypermultiplets we are confronted with the more difficult task of reducing a quaternionic manifold to a Kähler-Hodge one.

Note that, differently from what done in the preceeding sections, where we discussed only ungauged theories, the present reduction is given at the level of the complete $N = 2$ gauged theory.

In the first two subsections we begin to analyze the reduction in the vector multiplet sector, where much of the special geometry relations are needed. In subsection 6.3 we analyze the reduction in the hypermultiplet sector. In both cases the geometrical approach discussed in section 3 will be essential for the discussion. The other subsections are devoted

JHEP03(2002)025

N'	multiplet	max spin	multiplicity
8	$(g_{\mu\nu}, \psi_{A\mu} T_{AB\|\mu\nu}, \chi_{ABC}, P_{ABCD})$	2	1
6	$(g_{\mu\nu}, \psi_{a\mu} T_{ab\|\mu\nu}, T_{ij\|\mu\nu}, \chi_{abc}, \chi_{aij}, P_{abcd}, P_{abij})$	2	1
	$(\psi_{i\mu} T_{ai\|\mu\nu}, \chi_{abi}, P_{abci})$	3/2	2
5	$(g_{\mu\nu}, \psi_{a\mu} T_{ab\|\mu\nu}, \chi_{abc}, \chi_{ijk}, P_{abcd}, P_{aijk})$	2	1
	$(\psi_{i\mu} T_{ai\|\mu\nu}, T_{ij\|\mu\nu}, \chi_{abi}, \chi_{aij}, P_{abci}, P_{abij})$	3/2	3
4	$(g_{\mu\nu}, \psi_{a\mu} T_{ab\|\mu\nu}, \chi_{abc}, P_{abcd}, P_{ijk\ell})$	2	1
	$(\psi_{i\mu} T_{ai\|\mu\nu}, \chi_{abi}, \chi_{ijk}, P_{abci}, P_{aijk})$	3/2	4
	$(T_{ij\|\mu\nu}, \chi_{aij}, P_{abij})$	1	6
3	$(g_{\mu\nu}, \psi_{a\mu} T_{ab\|\mu\nu}, \chi_{abc})$	2	1
	$(\psi_{i\mu} T_{ai\|\mu\nu}, \chi_{abi}, P_{abci}, P_{ijk\ell})$	3/2	5
	$(T_{ij\|\mu\nu}, \chi_{aij}, P_{abij}, P_{aijk})$	1	10
2	$(g_{\mu\nu}, \psi_{a\mu} T_{ab\|\mu\nu})$	2	1
	$(\psi_{i\mu} T_{ai\|\mu\nu}, \chi_{abi})$	3/2	6
	$(T_{ij\|\mu\nu}, \chi_{aij}, P_{abij})$	1	15
	(χ_{ijk}, P_{aijk})	1/2	10
1	$(g_{\mu\nu}, \psi_{a\mu})$	2	1
	$(\psi_{i\mu} T_{ai\|\mu\nu})$	3/2	7
	$(T_{ij\|\mu\nu}, \chi_{aij})$	1	21
	(χ_{ijk}, P_{aijk})	1/2	35

Table 3: Decomposition of $N = 8$ into $N = N'$ supergravity multiplet.

$G = E_{6(6)}$	$G_1 \times G_2$	$G \to G_1 \times G_2$
$N = 2$ (# vect. $= n_V + 1$)		
$n_V = 0$	$F_{4(4)}$	$27 \to 1 + 26$
$n_V = 14$	$\mathrm{SU}^*(6) \times \mathrm{SU}(2)$	$27 \to (15, 1) + (6, 2)$
$n_V = 8$	$\mathrm{SL}(3, \mathbb{C}) \times \mathrm{SU}(2, 1)$	$27 \to (3, 3', 1) + (1, 3, 3') + (3', 1, 3)$
$n_V = 5$	$\mathrm{SL}(3, \mathbb{R}) \times G_{2(2)}$	$27 \to (6, 1) + (3, 7)$

Table 4: Duality reduction in $D = 5$.

to a careful analysis of the implications of the gauging, to the reduction of the scalar potential and to the discussion of some explicit examples.

The reduction is obtained by truncating the spin 3/2 multiplet containing the second gravitino $\psi_{\mu 2}$ and the graviphoton.

Here and in the following we use the notations both for $N = 2$ and $N = 1$ supergravity as given in reference [45], the only differences being that we use here world indices $\mathcal{I}, \overline{\mathcal{I}} = 1, \ldots, n_V$ and boldfaced gauge indices $\mathbf{\Lambda} = 0, 1, \ldots, n_V$ for quantities in the $N = 2$ vector multiplets (since we want to reserve the notation Λ and $i, \bar{\imath}$ for the indices of the reduced $N = 1$ theory) and that the holomorphic matrix appearing in the kinetic term of the vectors in the $N = 1$ theory will be renamed as follows:

$$\overline{\mathcal{N}}_{\Lambda\Sigma}(z^i) \equiv f_{\Lambda\Sigma}(z^i) \,. \tag{6.1}$$

JHEP03(2002)025

Let us write down the supersymmetry transformation laws of the $N = 2$ theory, up to 3-fermions terms [5]:

Supergravity transformation rules of the (left-handed) Fermi fields:

$$\delta\,\psi_{A\mu} = \widehat{\nabla}_\mu\,\epsilon_A + \left(\mathrm{i}\,g\,S_{AB}\eta_{\mu\nu} + \epsilon_{AB}T^-_{\mu\nu}\right)\gamma^\nu\epsilon^B \tag{6.2}$$
$$\delta\,\lambda^{\mathcal{I}A} = \mathrm{i}\,\nabla_\mu\,z^{\mathcal{I}}\gamma^\mu\epsilon^A + G^{-\mathcal{I}}_{\mu\nu}\gamma^{\mu\nu}\epsilon_B\epsilon^{AB} + gW^{\mathcal{I}AB}\epsilon_B \tag{6.3}$$
$$\delta\,\zeta_\alpha = \mathrm{i}\,\mathcal{U}^{B\beta}_u\,\nabla_\mu\,q^u\,\gamma^\mu\epsilon^A\epsilon_{AB}\,C_{\alpha\beta} + gN^A_\alpha\,\epsilon_A\,, \tag{6.4}$$

where:

$$\widehat{\nabla}_\mu\,\epsilon_A = \mathcal{D}_\mu\epsilon_A + \widehat{\omega}_{\mu|A}{}^B\epsilon_B + \widehat{\mathcal{Q}}_\mu\epsilon_A \tag{6.5}$$

and the SU(2) and U(1) 1-form "gauged" connections are respectively given by:

$$\widehat{\omega}_A{}^B = \omega_A{}^B + g_{(\mathbf{\Lambda})}\,A^{\mathbf{\Lambda}}\,P^x_{\mathbf{\Lambda}}\,(\sigma^x)_A{}^B\,, \tag{6.6}$$
$$\widehat{\mathcal{Q}} = \mathcal{Q} + g_{(\mathbf{\Lambda})}\,A^{\mathbf{\Lambda}}\,P^0_{\mathbf{\Lambda}}\,, \tag{6.7}$$
$$\mathcal{Q} = -\frac{\mathrm{i}}{2}\left(\partial_{\mathcal{I}}\mathcal{K}dz^{\mathcal{I}} - \partial_{\overline{\mathcal{I}}}\mathcal{K}d\overline{z}^{\overline{\mathcal{I}}}\right) \tag{6.8}$$

$\omega_A{}^B$, $\mathcal{Q}$ are the SU(2) and U(1) connections of the ungauged theory. Moreover we have:

$$\begin{aligned}\nabla_\mu z^{\mathcal{I}} &= \partial_\mu z^{\mathcal{I}} + g_{(\mathbf{\Lambda})}A^{\mathbf{\Lambda}}_\mu k^{\mathcal{I}}_{\mathbf{\Lambda}}\\ \nabla_\mu q^u &= \partial_\mu q^u + g_{(\mathbf{\Lambda})}A^{\mathbf{\Lambda}}_\mu k^u_{\mathbf{\Lambda}}\,.\end{aligned} \tag{6.9}$$

Supergravity transformation rules of the Bose fields:

$$\delta\,V^a_\mu = -\mathrm{i}\,\overline{\psi}_{A\mu}\,\gamma^a\,\epsilon^A - \mathrm{i}\,\overline{\psi}^A_\mu\,\gamma^a\,\epsilon_A \tag{6.10}$$
$$\delta\,A^{\mathbf{\Lambda}}_\mu = 2\overline{L}^{\mathbf{\Lambda}}\overline{\psi}_{A\mu}\epsilon_B\epsilon^{AB} + 2L^{\mathbf{\Lambda}}\overline{\psi}^A_\mu\epsilon^B\epsilon_{AB} + \mathrm{i}\,f^{\mathbf{\Lambda}}_{\mathcal{I}}\,\overline{\lambda}^{\mathcal{I}A}\gamma_\mu\epsilon^B\,\epsilon_{AB} + \mathrm{i}\,\overline{f}^{\mathbf{\Lambda}}_{\overline{\mathcal{I}}}\,\overline{\lambda}^{\overline{\mathcal{I}}}_A\gamma_\mu\epsilon_B\,\epsilon^{AB} \tag{6.11}$$
$$\delta\,z^{\mathcal{I}} = \overline{\lambda}^{\mathcal{I}A}\epsilon_A \tag{6.12}$$
$$\delta\,z^{\overline{\mathcal{I}}} = \overline{\lambda}^{\overline{\mathcal{I}}}_A\epsilon^A \tag{6.13}$$
$$\delta\,q^u = \mathcal{U}^u_{\alpha A}\left(\overline{\zeta}^\alpha\epsilon^A + C^{\alpha\beta}\epsilon^{AB}\overline{\zeta}_\beta\epsilon_B\right). \tag{6.14}$$

Here $T^-_{\mu\nu}$ appearing in the supersymmetry transformation law of the $N = 2$ left-handed gravitini is the "dressed" graviphoton defined as:

$$T^-_{\mu\nu} \equiv 2\mathrm{i}\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}}L^{\mathbf{\Sigma}}F^{\mathbf{\Lambda}-}_{\mu\nu} \tag{6.15}$$

while

$$G^{\mathcal{I}-}_{\mu\nu} = -g^{\mathcal{I}\overline{\mathcal{J}}}\overline{f}^{\mathbf{\Gamma}}_{\overline{\mathcal{J}}}\mathrm{Im}\mathcal{N}_{\mathbf{\Gamma\Lambda}}F^{\mathbf{\Lambda}-}_{\mu\nu} \tag{6.16}$$

are the "dressed" field strengths of the vectors inside the vector multiplets. Moreover the fermionic shifts S_{AB}, $W^{\mathcal{I}\,AB}$ and N^A_α are given in terms of the prepotentials and Killing vectors of the quaternionic geometry (suitably dressed with special geometry data) and of

JHEP03(2002)025

the special geometry Killing vectors, as follows:

$$S_{AB} = \mathrm{i}\frac{1}{2}P_{AB\,\mathbf{\Lambda}}\,L^{\mathbf{\Lambda}} \equiv \mathrm{i}\frac{1}{2}P^x_{\mathbf{\Lambda}}\sigma^x_{AB}L^{\mathbf{\Lambda}} \tag{6.17}$$

$$W^{\mathcal{I}\,AB} = \mathrm{i}P^{AB}_{\mathbf{\Lambda}}\,g^{\mathcal{I}\overline{\mathcal{J}}}f^{\mathbf{\Lambda}}_{\overline{\mathcal{J}}} + \epsilon^{AB}k^{\mathcal{I}}_{\mathbf{\Lambda}}\overline{L}^{\mathbf{\Lambda}} \tag{6.18}$$

$$N^A_\alpha = 2\mathcal{U}^A_{\alpha u}\,k^u_{\mathbf{\Lambda}}\overline{L}^{\mathbf{\Lambda}} \tag{6.19}$$

$$N^\alpha_A = -2\mathcal{U}^\alpha_{Au}\,k^u_{\mathbf{\Lambda}}L^{\mathbf{\Lambda}}\,. \tag{6.20}$$

We recall that the Killing vectors $k^{\mathcal{I}}_{\mathbf{\Lambda}}$ and $k^u_{\mathbf{\Lambda}}$ are related to the prepotentials by:

$$\begin{aligned} k^{\mathcal{I}}_{\mathbf{\Lambda}} &= \mathrm{i}g^{\mathcal{I}\overline{\mathcal{J}}}\partial_{\overline{\mathcal{J}}}P^0_{\mathbf{\Lambda}} \\ k^u_{\mathbf{\Lambda}} &= \frac{1}{6\lambda^2}\Omega^{x|vu}\nabla_v P^x_{\mathbf{\Lambda}}\,; \qquad \lambda = -1\,, \end{aligned} \tag{6.21}$$

where Ω^x_{uv} is the SU(2)-valued 2-form defined in section 6.3 below, and that the prepotential $P^0_{\mathbf{\Lambda}}$ satisfies:

$$P^0_{\mathbf{\Lambda}}L^{\mathbf{\Lambda}} = P^0_{\mathbf{\Lambda}}\overline{L}^{\mathbf{\Lambda}} = 0\,. \tag{6.22}$$

Since we are going to compare the $N = 2$ reduced theory with the standard $N = 1$ supergravity, we also quote the supersymmetry transformation laws of the latter theory [6, 46]. We have, up to 3-fermions terms:

$N = 1$ **transformation laws:**

$$\delta\psi_{\bullet\mu} = \mathcal{D}_\mu\epsilon_\bullet + \widehat{Q}_\mu\epsilon_\bullet + \mathrm{i}L(z,\overline{z})\gamma_\mu\varepsilon^\bullet \tag{6.23}$$

$$\delta\chi^i = \mathrm{i}\left(\partial_\mu z^i + g_{(\Lambda)}A^\Lambda_\mu k^i_\Lambda\right)\gamma^\mu\varepsilon_\bullet + N^i\varepsilon_\bullet \tag{6.24}$$

$$\delta\lambda^\Lambda_\bullet = \mathcal{F}^{(-)\Lambda}_{\mu\nu}\gamma^{\mu\nu}\varepsilon_\bullet + \mathrm{i}D^\Lambda\varepsilon_\bullet \tag{6.25}$$

$$\delta V^a_\mu = -\mathrm{i}\psi_\bullet\gamma_\mu\varepsilon^\bullet + \text{h.c.} \tag{6.26}$$

$$\delta A^\Lambda_\mu = \mathrm{i}\frac{1}{2}\overline{\lambda}^\Lambda_\bullet\gamma_\mu\varepsilon^\bullet + \text{h.c.} \tag{6.27}$$

$$\delta z^i = \overline{\chi}^i\varepsilon_\bullet\,, \tag{6.28}$$

where $\widehat{\mathcal{Q}}$ is defined in a way analogous to the $N = 2$ definition (6.7) and:

$$L(z,\overline{z}) = W(z)\,e^{\frac{1}{2}\mathcal{K}_V(z,\overline{z})} \tag{6.29}$$

$$N^i = 2\,g^{i\overline{\jmath}}\,\nabla_{\overline{\jmath}}\overline{L} \tag{6.30}$$

$$D^\Lambda = -2(\mathrm{Im}f_{\Lambda\Sigma})^{-1}P_\Sigma(z,\overline{z}) \tag{6.31}$$

and $W(z), \mathcal{K}_{(1)}(z,\overline{z}), P_\Sigma(z,\overline{z}), f_{\Lambda\Sigma}(z)$ are the superpotential, Kähler potential, Killing prepotential and vector kinetic matrix respectively [46, 6, 47]. Note that for the gravitino and gaugino fields we have denoted by a lower (upper) dot left-handed (right-handed) chirality. For the spinors of the chiral multiplets χ, instead, left-handed (right-handed) chirality is encoded via an upper holomorphic (antiholomorphic) world index $(\chi^i, \chi^{\overline{\imath}})$.

The supersymmetric lagrangians which are left invariant by these transformation laws are given in appendix G.

JHEP03(2002)025

To perform the truncation we set A=1 and 2 successively, putting $\psi_{2\mu} = \epsilon_2 = 0$, and we get from eq. (6.2):

$$\delta\,\psi_{1\mu} = \mathcal{D}_\mu \epsilon_1 - \widehat{\mathcal{Q}}_\mu \epsilon_1 - \widehat{\omega}_{\mu|1}{}^{1} \epsilon_1 \; + \mathrm{i}\, g\, S_{11} \eta_{\mu\nu} \gamma^\nu \epsilon^1 \,, \tag{6.32}$$

where $\mathcal{D}$ denotes the Lorentz covariant derivative (on the spinors, $\mathcal{D}_\mu = \partial_\mu - \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}$), while, for consistency:

$$\delta\,\psi_{2\mu} \equiv 0 = -\widehat{\omega}_{\mu|2}{}^{1}\epsilon_1 + \left(\mathrm{i}\, g\, S_{21}\eta_{\mu\nu} - T^-_{\mu\nu}\right)\gamma^\nu \epsilon^1 \,. \tag{6.33}$$

For a consistent truncation in the ungauged case we must set to zero the graviphoton:

$$T^- = T_{\mathbf{\Lambda}} F^{-\mathbf{\Lambda}} = 0 \,, \tag{6.34}$$

where

$$T_{\mathbf{\Sigma}} \equiv 2\mathrm{i}\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}} L^{\mathbf{\Lambda}} \tag{6.35}$$

is the projector on the graviphoton [48], and the component ${\omega_1}^2$ of the SU(2) connection 1-form:

$$\omega_1^2 = 0 \,. \tag{6.36}$$

In the gauged case we have the further constraints:

$$S_{21} = \frac{\mathrm{i}}{2} P^x_{\mathbf{\Lambda}} (\sigma^x)_{12} L^{\mathbf{\Lambda}} = \frac{\mathrm{i}}{2} P^3_{\mathbf{\Lambda}} L^{\mathbf{\Lambda}} = 0 \,, \tag{6.37}$$

$$\widehat{\omega}_1{}^2 = \omega_1{}^2 + g_{(\mathbf{\Lambda})} A^{\mathbf{\Lambda}} P^x_{\mathbf{\Lambda}} (\sigma^x)_1{}^2 \equiv g_{(\mathbf{\Lambda})} A^{\mathbf{\Lambda}} P^x_{\mathbf{\Lambda}} (\sigma^x)_1{}^2 = 0 \,, \tag{6.38}$$

while no further restriction comes from (6.7) since the form of the gauged U(1) connection should not change in the reduced theory.

Comparing (6.23) with (6.32), we learn that we must identify:

$$\begin{aligned} \psi_{1\mu} &\equiv \psi_{\bullet\mu} \\ \epsilon_1 &\equiv \epsilon_\bullet \,. \end{aligned} \tag{6.39}$$

Furthermore, $g\, S_{11} = \frac{\mathrm{i}}{2} g_{(\mathbf{\Lambda})} P^x_{\mathbf{\Lambda}} (\sigma^x)_{11} L^{\mathbf{\Lambda}}$ must be identified with the superpotential of the $N = 1$ theory, that is to the covariantly holomorphic section L of the $N = 1$ Kähler-Hodge manifold. Therefore we have [12]–[22]:

$$L(q, z, \overline{z}) = \frac{\mathrm{i}}{2} g_{(\mathbf{\Lambda})} P^x_{\mathbf{\Lambda}} (\sigma^x)_{11} L^{\mathbf{\Lambda}} = \frac{\mathrm{i}}{2} g_{(\mathbf{\Lambda})} \left(P^1_{\mathbf{\Lambda}} - \mathrm{i} P^2_{\mathbf{\Lambda}}\right) L^{\mathbf{\Lambda}} \,. \tag{6.40}$$

We will show in the following (section 6.4) that, after consistent reduction of the special-Kähler manifold $\mathcal{M}^{SK}$ and of the quaternionic σ-model $\mathcal{M}^Q$, L will in fact be a covariantly holomorphic function of the Kähler coordinates w^s of the reduced manifold $\mathcal{M}^{KH} \subset \mathcal{M}^Q$ and of some subset $z^i \in \mathcal{M}_R$ of the scalars $z^{\mathcal{I}}$ of the $N = 2$ special-Kähler manifold $\mathcal{M}^{SK}$.

The condition on the graviphoton $T^- = 0$ will be analyzed in subsection 6.1, while the condition ${\omega_1}^2 = 0$ will be discussed in section 6.3 and the constraints appearing in the gauged theory will be analyzed in section 6.4.

Here and in the following we will denote by $\mathcal{M}^{SK}$ and $\mathcal{M}^Q$ the special-Kähler and quaternionic manifolds of the $N = 2$ theory while the special-Kähler and Kähler-Hodge manifolds obtained by reduction of $\mathcal{M}^{SK}$ and $\mathcal{M}^Q$ will be denoted by $\mathcal{M}_R$ and $\mathcal{M}^{KH}$ respectively.

JHEP03(2002)025

6.1 Reduction of the $N = 2$ vector multiplet sector

Let us now consider the gaugino transformation laws. When $\epsilon_2 = 0$ we get:

$$\delta\,\lambda^{\mathcal{I}1} = \mathrm{i}\,\nabla_\mu\, z^{\mathcal{I}}\gamma^\mu\epsilon^1 + W^{\mathcal{I}11}\epsilon_1 \tag{6.41}$$
$$\delta\,\lambda^{\mathcal{I}2} = -G^{-\mathcal{I}}_{\mu\nu}\gamma^{\mu\nu}\epsilon_1 + g\,W^{\mathcal{I}21}\epsilon_1\,, \tag{6.42}$$

where, using (6.18)

$$W^{\mathcal{I}21} = \mathrm{i}P^3_{\mathbf{\Lambda}}\, g^{\mathcal{I}\overline{\mathcal{J}}} f^{\mathbf{\Lambda}}_{\overline{\mathcal{J}}} - k^{\mathcal{I}}_{\mathbf{\Lambda}}\overline{L}^{\mathbf{\Lambda}} \tag{6.43}$$
$$W^{\mathcal{I}11} = \mathrm{i}P^{11}_{\mathbf{\Lambda}} g^{\mathcal{I}\overline{\mathcal{J}}} f^{\mathbf{\Lambda}}_{\overline{\mathcal{J}}} = \left(P^2_{\mathbf{\Lambda}} - \mathrm{i}P^1_{\mathbf{\Lambda}}\right) g^{\mathcal{I}\overline{\mathcal{J}}} f^{\mathbf{\Lambda}}_{\overline{\mathcal{J}}}\,. \tag{6.44}$$

From eqs. (6.41) and (6.42) we immediately see that the spinors $\lambda^{\mathcal{I}1}$ transform into the scalars $z^{\mathcal{I}}$ (and should therefore give rise to $N = 1$ chiral multiplets) while the spinors $\lambda^{\mathcal{I}2}$ transform into the matter vectors field strengths $G^{-\mathcal{I}}_{\mu\nu}$ (and should then be identified with the gauginos of the $N = 1$ vector multiplets).

However, before entering the details of the identification, we have to discuss the implications of putting to zero the graviphoton T^-, eq. (6.34). We observe that this condition gives a constraint on the scalar and vector content of the $N = 1$ reduced theory, that is on the number of chiral and vector multiplets which are retained after truncation.

Now, since the graviphoton projector $T_{\mathbf{\Lambda}}$ (6.35) is a scalar field dependent quantity, the request that eq. (6.34) is verified all over the manifold can be trivially realized either by setting to zero all the scalars $z^{\mathcal{I}}$ and the graviphoton A^0_μ, which implies on the symplectic section $L^{\mathbf{\Lambda}} \Rightarrow (L^0 = 1; L^\Lambda = 0, \Lambda = 1, \ldots n_V)$, or, alternatively, by truncating out all the vectors $A^{\mathbf{\Lambda}}$, leaving an $N = 1$ theory with only chiral matter content.

There is however a more interesting and non trivial way to satisfy eq (6.34), by imposing a suitable constraint on the set of vectors and of scalar sections which can be retained in the reduction. Indeed, if we decompose the index $\mathbf{\Lambda}$ labelling the vectors into two disjoint sets $\mathbf{\Lambda} \Rightarrow (\Lambda, X), \Lambda = 1, \ldots, n'_V = n_V - n_C; X = 0, 1, \ldots, n_C$, we may satisfy the relation (6.34) as an "orthogonality relation" between the subset Λ running on the retained vectors and the subset X running on the retained scalar sections. That is we set:

$$\begin{aligned} F^X_{\mu\nu} &= 0\,; \\ \mathrm{Im}\mathcal{N}_{\Lambda\mathbf{\Sigma}}L^{\mathbf{\Sigma}} &= T_\Lambda = 0\,. \end{aligned} \tag{6.45}$$

We note that if we delete the electric field strengths F^{-X} we must also delete their magnetic counterpart

$$\mathcal{G}^-_X = \overline{\mathcal{N}}_{XY}F^{-Y} + \overline{\mathcal{N}}_{X\Sigma}F^{-\Sigma} = 0 \tag{6.46}$$

so that we must also impose

$$\mathcal{N}_{X\Sigma} = 0\,. \tag{6.47}$$

Then, the constraint (6.45) reduces to

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}L^\Sigma = 0 \tag{6.48}$$

JHEP03(2002)025

which implies

$$L^{\Sigma} = 0 \tag{6.49}$$

since the vector-kinetic matrix $\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}$ has to be invertible.

Note that conditions (6.48) and (6.49) imply a reduction of the $N = 2$ scalar manifold $\mathcal{M}^{SK} \to \mathcal{M}_R$, since it says that some coordinate dependent sections on $\mathcal{M}^{SK}$ have to be zero in the reduced theory.

Let us decompose the world indices $\mathcal{I}$ of the $N = 2$ special-Kähler σ-model as follows: $\mathcal{I} \Rightarrow (i, \alpha)$, with $i = 1, \ldots, n_C$, $\alpha = 1, \ldots, n'_V = n_V - n_C$, where n_C and n'_V are respectively the number of chiral and vector multiplets of the reduced $N = 1$ theory while n_V is the number of $N = 2$ vector multiplets.

Then from eq (6.53) it follows that the metric on $\mathcal{M}_R$ is pulled back to the following form [5, 48]:

$$g_{i\bar{\jmath}} = -2 f_i^X \mathrm{Im}\mathcal{N}_{XY} \overline{f}_{\bar{\jmath}}^Y \,. \tag{6.50}$$

To examine further the implications of the reduction of the special-Kähler manifold to the submanifold $\mathcal{M}_R$, it is convenient to write the special geometry objects using flat indices. We then define a set of kählerian vielbeins $P^{\widehat{I}} = P^{\widehat{I}}_{\mathcal{I}} dz^{\mathcal{I}}$ on $\mathcal{M}^{SK}$ together with their complex conjugates. Performing the reduction, they decompose as: $P^{\widehat{I}} \Rightarrow (P^I, P^A)$, where I and A are flat indices in the submanifold $\mathcal{M}_R$ and on its orthogonal complement respectively. By an appropriate choice of coordinates, we call z^i the coordinates on $\mathcal{M}_R$, z^α the coordinates on the orthogonal complement. Then we may set $P^I_\alpha = 0, P^A_i = 0$, so that the metric $g_{\mathcal{I}\overline{\mathcal{J}}} = P^{\widehat{I}}_{\mathcal{I}} \overline{P}^{\widehat{I}}_{\overline{\mathcal{J}}}$ has only components $g_{i\bar{\jmath}}, g_{\alpha\bar{\beta}}$, while $g_{i\bar{\alpha}} = 0$.

Then, if we decompose the gauginos $\lambda^{\mathcal{I}2} \Rightarrow (\lambda^{i2}, \lambda^{\alpha 2})$, the above truncation implies, by supersymmetry, $\lambda^{i2} = 0$ and, for consistency,

$$\delta\,\lambda^{i2} = -G^{-i}_{\mu\nu}\gamma^{\mu\nu}\epsilon_1 \,+\, g\,W^{i21}\epsilon_1 = 0\,. \tag{6.51}$$

Setting $G^{-i}_{\mu\nu} = 0$ gives:

$$G^{-i}_{\mu\nu} = -g^{i\overline{\mathcal{J}}} \nabla_{\overline{\mathcal{J}}} \overline{L}^{\boldsymbol{\Lambda}} \mathrm{Im}\mathcal{N}_{\boldsymbol{\Lambda\Sigma}} F^{-\boldsymbol{\Sigma}}_{\mu\nu} = -g^{i\bar{\jmath}} \nabla_{\bar{\jmath}} \overline{L}^{\Lambda} \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} F^{-\Sigma}_{\mu\nu} = 0 \tag{6.52}$$

implying

$$\nabla_{\bar{\jmath}} \overline{L}^{\Lambda} = \overline{f}^{\Lambda}_{\bar{\jmath}} = 0\,. \tag{6.53}$$

Moreover, $W^{i21} = 0$ implies:

$$P^3_X = 0\,, \qquad k^i_X = 0\,. \tag{6.54}$$

Note that the integrability condition of eq. (6.53) is:

$$\nabla_i \nabla_j L^{\Lambda} = \mathrm{i} C_{ij\mathcal{K}} g^{\mathcal{K}\overline{\mathcal{K}}} \nabla_{\overline{\mathcal{K}}} \overline{L}^{\Lambda} = \mathrm{i} C_{ijk} g^{k\bar{k}} \nabla_{\bar{k}} \overline{L}^{\Lambda} + \mathrm{i} C_{ij\alpha} g^{\alpha\bar{\alpha}} \nabla_{\bar{\alpha}} \overline{L}^{\Lambda} = 0\,, \tag{6.55}$$

where C_{ijk} is the 3-index symmetric tensor appearing in the equations defining the special geometry (see e.g. ref. [48, 5]).

Since the first term on the r.h.s. of eq. (6.55) is zero on $\mathcal{M}_R$ (eq. (6.53)), eq. (6.55) is satisfied by imposing:

$$C_{ij\alpha} = 0 \tag{6.56}$$

so that only the $N = 2$ special-Kähler manifolds satisfying the constraint (6.56) are suitable for reduction.

JHEP03(2002)025

Note that, since $C_{ij\alpha}$ is defined as a symplectic scalar product [3, 8, 9, 5] in terms of the symplectic section $U = (L^{\mathbf{\Lambda}}, M_{\mathbf{\Lambda}})$:

$$C_{ij\alpha} = \langle \nabla_i \nabla_j U, \nabla_\alpha U \rangle \,, \tag{6.57}$$

it follows that

$$C_{ij\alpha} = 0 \Rightarrow \nabla_\alpha U = 0 \Rightarrow \nabla_\alpha L^X = 0 \,, \qquad \nabla_\alpha M_X = 0 \,. \tag{6.58}$$

The same constraint (6.56) can also be retrieved by looking at the integrability conditions of the $N = 2$ special geometry as given in [43]. The relevant ones for our discussion are the following:

$$\begin{aligned} \nabla P^{\widehat{I}} &= dP^{\widehat{I}} + \mathrm{i}\mathcal{Q} \wedge P^{\widehat{I}} + \omega^{\widehat{I}}{}_{\widehat{J}} \wedge P^{\widehat{J}} = 0 \\ R^{\overline{\widehat{J}}}{}_{\overline{\widehat{I}}} &\equiv (d\omega + \omega \wedge \omega)^{\overline{\widehat{J}}}{}_{\overline{\widehat{I}}} = P_{\overline{\widehat{I}}} \wedge \overline{P}^{\overline{\widehat{J}}} - \mathrm{i}K \delta^{\overline{\widehat{J}}}_{\overline{\widehat{I}}} - C^{\overline{\widehat{J}}}{}_{\widehat{L}} \wedge \overline{C}^{\widehat{L}}{}_{\overline{\widehat{I}}} \,, \end{aligned} \tag{6.59}$$

where $\mathcal{Q}$ is the Kähler connection 1-form, $K = d\mathcal{Q}$ is the Kähler 2-form, $\omega^{\widehat{I}}{}_{\widehat{J}}$ is the $\mathrm{SU}(n_V)$-Lie algebra valued connection and the 1-form $C^{\overline{\widehat{J}}}{}_{\widehat{L}}$ can be written in terms of the 3-world indices symmetric tensor $C_{\mathcal{I}\mathcal{J}\mathcal{K}}$, whose properties are given in ref. [43], via:

$$C^{\overline{\widehat{J}}}{}_{\widehat{L}} = P^{\overline{\widehat{J}}\mathcal{I}} P_{\widehat{L}}{}^{\mathcal{J}} C_{\mathcal{I}\mathcal{J}\mathcal{K}} dz^{\mathcal{K}} . \tag{6.60}$$

Let us restrict the previous equations to the submanifold $\mathcal{M}_R$. From the vanishing of the torsion, eq (6.59), we find:

$$\begin{aligned} \nabla P^I &= dP^I + \mathrm{i}\mathcal{Q} \wedge P^I + \omega^I{}_J \wedge P^J + \omega^I{}_A \wedge P^A = 0 \\ \nabla P^A &= dP^A + \mathrm{i}\mathcal{Q} \wedge P^A + \omega^A{}_J \wedge P^J + \omega^A{}_B \wedge P^B = 0 \,. \end{aligned} \tag{6.61}$$

With the same procedure illustrated in the general discussion of section 4 and in the example of section 6.1, we easily find that the vanishing of the torsion on $\mathcal{M}_R$ implies $\omega^I{}_A|_{\mathcal{M}_R} = 0$, from which it follows, taking into account the Frobenius theorem and the definition of $R^{\overline{\widehat{J}}}{}_{\overline{\widehat{I}}}$:

$$R^{\overline{J}}{}_{\overline{A}}|_{\mathcal{M}_R} = P_{\overline{A}} \wedge \overline{P}^{\overline{J}} - C^{\overline{J}}{}_I \wedge \overline{C}^I{}_{\overline{A}} - C^{\overline{J}}{}_B \wedge \overline{C}^B{}_{\overline{A}} = 0 \,. \tag{6.62}$$

Now, expanding the vielbein and the C-tensor along the differentials of the coordinates, we easily find

$$R^{\overline{J}}{}_{\overline{A}}|_{\mathcal{M}_R} = -\overline{P}^{\overline{J}i} P^\alpha_{\overline{A}} \left(C_{ijk} \overline{C}^j{}_{\alpha\ell} + C_{i\beta k} \overline{C}^\beta{}_{\alpha\ell} \right) dz^k \wedge dz^\ell = 0 \,, \tag{6.63}$$

where we have set to zero the terms in the external directions dz^α, and the C-terms containing both holomorphic and antiholomorphic indices, which are zero already because of the $N = 2$ special geometry properties [3, 5]. Again, we see that equation (6.63) is satisfied by imposing the same condition (6.56) on the special-Kähler manifold.

JHEP03(2002)025

From the analysis of the fermionic terms in the supersymmetry transformation laws of the fermions [49], it is possible to find a further condition on the C-tensor:

$$C_{\alpha\beta\gamma}|_{\mathcal{M}_R} = 0 \tag{6.64}$$

which, together with (6.56), implies

$$R^i{}_{\alpha\beta\gamma}|_{\mathcal{M}_R} = 0\,. \tag{6.65}$$

6.2 $N = 2$ vector multiplets $\longrightarrow N = 1$ matter multiplets

Let us now discuss the precise identification of the $N = 1$ matter multiplets obtained by reduction of the $N = 2$ vector multiplets.

From the above analysis we have found that the indices labelling $N = 1$ chiral and vector multiplets are not related anymore, as it was instead the case in the $N = 2$ theory.

As far as eq. (6.41) is concerned, we immediately see that, after reduction of the index $\mathcal{I}$ and comparison with the corresponding $N = 1$ formula (6.24), we can make the following identification:

$$\lambda^{i1} = \chi^i \tag{6.66}$$

$$gW^{i11} = N^i = \mathrm{i}g_{(X)}\left(P^1_X - \mathrm{i}P^2_X\right) g^{i\bar{\jmath}} f^X_{\bar{\jmath}} \tag{6.67}$$

that is we may interpret the λ^{i1} as n_C $N = 1$ chiral spinors belonging to $N = 1$ left-handed chiral multiplets (χ^i, z^i), $i = 1, \ldots, n_C$. It can be easily verified that the consistency condition

$$\lambda^{\alpha 1} = 0 \Rightarrow \delta\lambda^{\alpha 1} = 0 \tag{6.68}$$

gives

$$k^\alpha_\Lambda = 0 \tag{6.69}$$

using $f^X_\alpha = 0$.

Let us now discuss the $N = 1$ vector multiplets coming from the truncation.

The transformation law for the $n_V + 1$ vectors of the $N = 2$ theory (6.11) becomes, after truncation:

$$\delta A^\Lambda_\mu = -\mathrm{i}f^\Lambda_i \overline{\lambda}^{i2}\gamma_\mu\epsilon^1 - \mathrm{i}f^\Lambda_\alpha\overline{\lambda}^{\alpha 2}\gamma_\mu\epsilon^1 + \text{h.c.} = -\mathrm{i}f^\Lambda_\alpha\overline{\lambda}^{\alpha 2}\gamma_\mu\epsilon^1 + \text{h.c.} \tag{6.70}$$

$$\delta A^X_\mu = -\mathrm{i}f^X_i \overline{\lambda}^{i2}\gamma_\mu\epsilon^1 - \mathrm{i}f^X_\alpha\overline{\lambda}^{\alpha 2}\gamma_\mu\epsilon^1 + \text{h.c.} = 0\,, \tag{6.71}$$

where in (6.70) we have used (6.53). Eq. (6.71) is consistently zero if we put

$$\lambda^{i2} = 0 \tag{6.72}$$

since $f^X_\alpha|_{\mathcal{M}_R} = \nabla_\alpha L^X|_{\mathcal{M}_R} = 0$.[8]

[8] This follows by looking at the expression of the $N = 2$ Kähler metric [5]

$$g_{\mathcal{I}\bar{\mathcal{J}}} = -2\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}} f^{\mathbf{\Lambda}}_{\mathcal{I}} f^{\mathbf{\Sigma}}_{\bar{\mathcal{J}}} \tag{6.73}$$

by requiring that its mixed component $g_{i\bar{\alpha}}$ is zero. Indeed, after reduction we get

$$0 = g_{i\bar{\alpha}} = -2\mathrm{Im}\mathcal{N}_{XY} f^X_i f^Y_{\bar{\alpha}} \tag{6.74}$$

implying $f^Y_{\bar{\alpha}} = 0$.

JHEP03(2002)025

We note that while the gauge index $\mathbf{\Lambda}$ of the $N=2$ gaugino runs over n_V+1 values (because of the presence of the graviphoton) the indices Λ and α take only $n'_V \leq n_V$ values. In particular, the index of the graviphoton A^0 belongs to the orthogonal subset $X=0,1,\ldots,n_C$, so that the graviphoton is automatically projected out.

To match the corresponding $N=1$ formula (6.27) we have to set:

$$\lambda^\Lambda_\bullet \equiv -2f^\Lambda_\alpha \lambda^{\alpha 2}\,. \tag{6.75}$$

Now, we observe that we may trade the gaugino world index $\mathcal{I}=1,\ldots,n_V$ with a vector index $\mathbf{\Lambda}$ already at the level of the $N=2$ theory, by defining

$$\lambda^{\mathbf{\Lambda}A} \equiv -2f^{\mathbf{\Lambda}}_{\mathcal{I}} \lambda^{\mathcal{I}A}\,. \tag{6.76}$$

Here the gauge index $\mathbf{\Lambda}$ of the $N=2$ gauginos runs over n_V+1 values (because of the presence of the graviphoton) while the index $\mathcal{I}$ takes only n_V values. The extra gaugino, say λ^0, is actually spurious, since $\lambda^{\mathbf{\Lambda}A}$ satisfies:

$$T_{\mathbf{\Lambda}}\lambda^{\mathbf{\Lambda}A} = -2T_{\mathbf{\Lambda}} f^{\mathbf{\Lambda}}_{\mathcal{I}}\lambda^{\mathcal{I}A} = 0\,, \tag{6.77}$$

where

$$T_{\mathbf{\Lambda}} \equiv 2\mathrm{i}\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}}L^{\mathbf{\Sigma}}\,, \tag{6.78}$$

due to the special geometry relation

$$\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}}L^{\mathbf{\Lambda}}f^{\mathbf{\Sigma}}_{\mathcal{I}} = 0\,. \tag{6.79}$$

Note that $T_{\mathbf{\Lambda}}$ is the projector on the graviphoton field-strength, according to equation (6.15) [48].

Using special geometry, one can see that the transformation law for the $N=2$ gaugini (6.3) can be rewritten in terms of the $\lambda^{\mathbf{\Lambda}A}$, up to 3-fermions terms, as:

$$\delta\lambda^{\mathbf{\Lambda}A} = P^{\mathbf{\Lambda}}{}_{\mathbf{\Sigma}}F^{-\mathbf{\Sigma}}_{\mu\nu}\gamma^{\mu\nu}\epsilon^{AB}\epsilon_B - 2\mathrm{i}U^{\mathbf{\Lambda\Sigma}}\left(P^0_{\mathbf{\Sigma}}\epsilon^{AB} + P^{AB}_{\mathbf{\Sigma}}\right)\epsilon_B\,, \tag{6.80}$$

where $P^{\mathbf{\Lambda}}{}_{\mathbf{\Sigma}}$ is the projector on the matter-vector field strengths and $U^{\mathbf{\Lambda\Sigma}}$ a tensor of special geometry. They are defined below in equations (6.84), (6.83). The derivation of formula (6.80) is given in appendix C.

The above formulae allow us to perform the reduction of the gaugino $\lambda^{\mathbf{\Lambda}2} = (\lambda^{\Lambda 2}, \lambda^{X2})$ straightforwardly. First of all, $\lambda^{X2} = f^X_i\lambda^{i2} = 0$ as follows from (6.72). Then, setting $A=2$ and $\mathbf{\Lambda}=\Lambda$, we have

$$\lambda^\Lambda_\bullet \equiv \lambda^{\Lambda 2} = -2f^\Lambda_{\mathcal{I}}\lambda^{\mathcal{I}2} = -2f^\Lambda_\alpha\lambda^{\alpha 2} \tag{6.81}$$

since in the reduced theory $f^\Lambda_i = 0$, and then:

$$\delta\lambda^\Lambda_\bullet = P^\Lambda{}_\Sigma F^{-\Sigma}_{\mu\nu}\gamma^{\mu\nu}\epsilon_\bullet - 2\mathrm{i}U^{\Lambda\Sigma}\left(P^0_\Sigma + P^3_\Sigma\right)\epsilon_\bullet\,. \tag{6.82}$$

Let us now apply the following relations of special geometry [48] to the present reduction:

$$U^{\mathbf{\Lambda\Sigma}} \equiv f^{\mathbf{\Lambda}}_{\mathcal{I}}g^{\mathcal{I}\overline{\mathcal{J}}}f^{\mathbf{\Sigma}}_{\overline{\mathcal{J}}} = -\frac{1}{2}\left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{\mathbf{\Lambda\Sigma}} - \overline{L}^{\mathbf{\Lambda}}L^{\mathbf{\Sigma}} \tag{6.83}$$

$$P^{\mathbf{\Lambda}}{}_{\mathbf{\Sigma}} \equiv -2U^{\mathbf{\Lambda\Gamma}}\mathrm{Im}\mathcal{N}_{\mathbf{\Sigma\Gamma}} = \delta^{\mathbf{\Lambda}}_{\mathbf{\Sigma}} - \mathrm{i}T_{\mathbf{\Sigma}}\overline{L}^{\mathbf{\Lambda}}\,. \tag{6.84}$$

JHEP03(2002)025

After decomposing the indices and using $f^X_\alpha|_{\mathcal{M}_R} = f^\Lambda_i|_{\mathcal{M}_R} = g_{i\overline{\alpha}}|_{\mathcal{M}_R} = 0$ we have [5]:

$$U^{\Lambda\Sigma} \equiv f^\Lambda_\alpha g^{\alpha\beta} f^\Sigma_\beta = -\frac{1}{2}\left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{\Lambda\Sigma} ; \tag{6.85}$$

$$U^{XY} \equiv f^X_i g^{i\overline{\jmath}} f^Y_{\overline{\jmath}} = -\frac{1}{2}\left[(\mathrm{Im}\mathcal{N})^{-1}\right]^{XY} - \overline{L}^X L^Y ; \tag{6.86}$$

$$P^\Lambda{}_\Sigma = (\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Gamma}\mathrm{Im}\mathcal{N}_{\Sigma\Gamma} = \delta^{\mathbf{\Lambda}}_{\mathbf{\Sigma}} ; \tag{6.87}$$

$$P^X{}_Y = = \delta^X_Y - \mathrm{i}T_Y\overline{L}^X . \tag{6.88}$$

Eq. (6.82) can then be rewritten as:

$$\delta\lambda^\Lambda_\bullet = \left[F^{-\Lambda}_{\mu\nu}\gamma^{\mu\nu} + \mathrm{i}\left(\mathrm{Im}\mathcal{N}^{-1}\right)^{\Lambda\Sigma}\left(P^0_\Sigma + P^3_\Sigma\right)\right]\epsilon_\bullet . \tag{6.89}$$

We observe that the prepotential P^0_Σ, which gives the special-Kähler manifold contribution to the D-term, can be given an explicit form in terms of $N = 2$ objects. Indeed, let us recall that $P^0_{\mathbf{\Sigma}}$ has the following general form, as shown in eq. (C.10) of appendix C:

$$P^0_{\mathbf{\Sigma}} = -2\mathrm{i}\mathrm{Im}\mathcal{N}_{\mathbf{\Sigma\Gamma}} f^{\mathbf{\Gamma}}_{\mathcal{I}} k^{\mathcal{I}}_{\mathbf{\Delta}} \overline{L}^{\mathbf{\Delta}} \tag{6.90}$$

which gives, after reduction:

$$P^0_\Sigma = -2\mathrm{i}\mathrm{Im}\mathcal{N}_{\Sigma\Gamma} f^\Gamma_\alpha k^\alpha_W \overline{L}^W . \tag{6.91}$$

On the other hand, using the following $N = 2$ special geometry property:

$$f^{\mathbf{\Gamma}}_{\mathcal{I}} k^{\mathcal{I}}_{\mathbf{\Delta}} = \mathrm{i}P^0_{\mathbf{\Delta}} L^{\mathbf{\Gamma}} - f^{\mathbf{\Gamma}}{}_{\mathbf{\Delta\Sigma}} L^{\mathbf{\Sigma}} \tag{6.92}$$

($f^{\mathbf{\Gamma}}{}_{\mathbf{\Delta\Sigma}}$ are the structure constant of the $N = 2$ gauge group $G^{(2)}$) by contracting with $\overline{L}^{\mathbf{\Delta}}$ and reducing it to the submanifold $\mathcal{M}_R$, we also find [5]:

$$P^0_\Sigma = 2\mathrm{i}\mathrm{Im}\mathcal{N}_{\Sigma\Gamma} f^\Gamma{}_{XY} \overline{L}^X L^Y . \tag{6.93}$$

In conclusion we get the final form of the gaugino transformation law for the $N = 1$ theory as:

$$\left(\delta\lambda^\Lambda_\bullet\right)_{N=1} = \mathcal{F}^{-\Lambda}_{\mu\nu}\gamma^{\mu\nu}\epsilon_\bullet + \mathrm{i}D^\Lambda\epsilon_\bullet , \qquad (\Lambda = 1, \ldots, n) , \tag{6.94}$$

where, in order to retrieve the transformation law (6.25) we have set

$$D^\Lambda \equiv (\mathrm{Im}\mathcal{N}^{-1})^{\Lambda\Sigma}\left(P^0_\Sigma + P^3_\Sigma\right) . \tag{6.95}$$

In order to show that equation (6.94) is the correct $N = 1$ transformation law of the gauginos we have still to prove that $\mathcal{N}_{\Lambda\Sigma}$ is an antiholomorphic function of the scalar fields z^i (as it is the case for an $N = 1$ theory), since the corresponding object of the $N = 2$ special geometry $\mathcal{N}_{\mathbf{\Lambda\Sigma}}$ is not antiholomorphic. For this purpose we observe that in $N = 2$ special geometry the following identity holds (at least when a $N = 2$ prepotential function exists[9])[48]:

$$\mathcal{N}_{\mathbf{\Lambda\Sigma}} = \overline{F}_{\mathbf{\Lambda\Sigma}} - 2\mathrm{i}\overline{T}_{\mathbf{\Lambda}}\overline{T}_{\mathbf{\Sigma}}(L^{\mathbf{\Gamma}}\mathrm{Im}F_{\mathbf{\Gamma\Delta}}L^{\mathbf{\Delta}}) , \tag{6.96}$$

where the matrix $F_{\mathbf{\Lambda\Sigma}}$ is holomorphic.

[9] In appendix D we will discuss the reduction with special coordinates.

JHEP03(2002)025

If we now reduce the indices $\mathbf{\Lambda\Sigma}$ we find:

$$\mathcal{N}_{\Lambda\Sigma} = \overline{F}_{\Lambda\Sigma} - 2\mathrm{i}\overline{T}_\Lambda \overline{T}_\Sigma (L^X \mathrm{Im} F_{XY} L^Y) \equiv \overline{F}_{\Lambda\Sigma} \tag{6.97}$$

since $T_\Lambda = 0$ is precisely the constraint (6.45). Therefore $\mathcal{N}_{\Lambda\Sigma}$ is antiholomorphic and the D-term (6.95) becomes:

$$D^\Lambda \equiv 2\mathrm{i} f_\alpha^\Lambda W^{\alpha 21} = -2(\mathrm{Im} f^{-1}(z^i))^{\Lambda\Sigma} \left(P^0_\Sigma + P^3_\Sigma \right), \tag{6.98}$$

where we have defined

$$F_{\Lambda\Sigma}(z^i) = \frac{1}{2} f_{\Lambda\Sigma}(z^i) \tag{6.99}$$

in order to match the normalization of the holomorphic kinetic matrix of the $N = 1$ theory appearing in eq. (6.31).

We observe that for choices of symplectic sections such that the function $F_{\mathbf{\Lambda\Sigma}}$ does not exist, the relation (6.97) does not hold, but still $\mathcal{N}_{\Lambda\Sigma}$ has to be antiholomorphic on $\mathcal{M}_R$. Un explicit example will be given in section 6.6.

As a final observation, we note that the above reduction on the indices of the $N = 2$ Killing vectors gives rise to $k^{\mathcal{I}}_{\mathbf{\Lambda}} \Rightarrow (k^i_\Lambda, k^\alpha_\Lambda, k^i_X, k^\alpha_X)$. The Killing vectors k^i_Λ gauge the isometries of the submanifold $\mathcal{M}_R$. On the other hand, k^α_Λ are zero on the submanifold, since they correspond to isometries orthogonal to $\mathcal{M}_R$; k^i_X are also zero because we have projected out the corresponding vectors. Finally, k^α_X are in general different form zero, and enter in the definition of P^0_Σ, eq. (6.91). These conclusions can be formally retrieved by analyzing the reduction of the special geometry identity [5]

$$2\mathrm{i} g_{\mathcal{I}\overline{\mathcal{J}}} k^{\mathcal{I}}_{[\mathbf{\Lambda}} k^{\overline{\mathcal{J}}}_{\mathbf{\Sigma}]} = f_{\mathbf{\Lambda\Sigma}}{}^{\mathbf{\Gamma}} P^0_{\mathbf{\Gamma}}\,. \tag{6.100}$$

One can easily verify that if we set $\mathbf{\Lambda} = \Lambda; \mathbf{\Sigma} = \Sigma$ then one retrieves the analogous of relation (6.100) on $\mathcal{M}_R$ provided we set

$$k^\alpha_\Lambda = 0\,; \qquad P^0_X = 0\,. \tag{6.101}$$

For $\mathbf{\Lambda} = \Lambda; \mathbf{\Sigma} = Y$ eq. (6.100) is identically satisfied provided we add to the previous condition the further constraint

$$k^i_X = 0\,. \tag{6.102}$$

Finally, when $\mathbf{\Lambda} = X; \mathbf{\Sigma} = Y$, eq. (6.100) reduces to the relation:

$$2\mathrm{i} g_{\alpha\overline{\beta}} k^\alpha_{[X} k^{\overline{\beta}}_{Y]} = f_{XY}{}^\Gamma P^0_\Gamma \tag{6.103}$$

which has to be satisfied all over the manifold $\mathcal{M}_R$.

6.3 Reduction of the hypermultiplet sector

Let us analyze the sector of the hypermultiplets when the reduction is implemented. The scalars of the hypermultiplets belong to a quaternionic manifold $\mathcal{M}^Q$. A quaternionic manifold $\mathcal{M}^Q$ has a holonomy group of the following type [50], [51], [11]:

$$\begin{aligned} \mathrm{Hol}(\mathcal{M}^Q) &= \mathrm{SU}(2) \otimes \mathcal{H} \quad \text{(quaternionic)} \\ \mathcal{H} &\subseteq \mathrm{Sp}(2n_H)\,. \end{aligned} \tag{6.104}$$

JHEP03(2002)025

Introducing flat indices $\{A, B, C = 1, 2\}$ $\{\alpha, \beta, \gamma = 1, \ldots, 2n_H\}$ that run, respectively, in the fundamental representations of SU(2) and Sp($2n_H$) (n_H is the number of hypermultiplets) we introduce the vielbein 1-form [5]

$$\mathcal{U}^{A\alpha} = \mathcal{U}^{A\alpha}_u(q) dq^u \tag{6.105}$$

such that

$$h_{uv} = \mathcal{U}^{A\alpha}_u \mathcal{U}^{B\beta}_v \mathbb{C}_{\alpha\beta} \epsilon_{AB} \,, \tag{6.106}$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat Sp($2n_H$) and Sp(2) $\sim$ SU(2) invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the SU(2)-connection $\omega^x (x = 1, 2, 3)$ and to the Sp($2n_H$)-Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\nabla \mathcal{U}^{A\alpha} \equiv d\mathcal{U}^{A\alpha} + \frac{i}{2} \omega^x \left(\sigma_x\right)_{\ B}^{A} \wedge \mathcal{U}^{B\alpha} + \Delta^{\alpha}_{\ \beta} \wedge \mathcal{U}^{A\beta} = 0 \,, \tag{6.107}$$

where $(\sigma^x)^{AB} = \epsilon^{AC} (\sigma^x)_C^{\ B}$ and $(\sigma^x)_A^{\ B}$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta} \,. \tag{6.108}$$

The supersymmetry transformation laws of the fields in the hypermultiplets are given in eq. (6.4) and (6.14), that we rewrite here using tangent-space indices for the quaternionic variation:

$$\begin{aligned} \mathcal{U}^{\alpha A}_u \delta\, q^u &= \overline{\zeta}^\alpha \epsilon^A + \mathbb{C}^{\alpha\beta} \epsilon^{AB} \overline{\zeta}_\beta \epsilon_B && (6.109)\\ \delta\, \zeta_\alpha &= \mathrm{i} \mathcal{U}^{B\beta}_u \nabla_\mu q^u \gamma^\mu \epsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta} + g\, N^A_\alpha \epsilon_A && (6.110)\\ \delta\, \zeta^\alpha &= \mathrm{i} \mathcal{U}^{A\alpha}_u \nabla_\mu q^u \gamma^\mu \epsilon_A + g\, N^\alpha_A \epsilon^A \,. && (6.111) \end{aligned}$$

Let us see what happens to equations (6.109), (6.110), (6.111), when the truncation is implemented.

First of all let us note that the scalars in $N = 1$ supergravity must lie in chiral multiplets, and have in general a Kähler-Hodge structure. It is therefore required that the holonomy of the quaternionic manifold be reduced:

$$\mathrm{Hol}\left(\mathcal{M}^Q\right) \subset \mathrm{SU}(2) \times \mathrm{Sp}(2n_H) \rightarrow \mathrm{Hol}\left(\mathcal{M}^{KH}\right) \subset \mathrm{U}(1) \times \mathrm{SU}(n) \,. \tag{6.112}$$

Therefore the SU(2) index $A = 1, 2$ and the Sp($2n_H$) index have to be decomposed accordingly. We set $\alpha \rightarrow (I, \dot{I}) \in \mathrm{U}(1) \times \mathrm{SU}(n_H) \subset \mathrm{Sp}(2n_H)$. Since the vielbein $\mathcal{U}^{A\alpha}$ satisfy the reality condition (6.108), we have, in $U(n_H)$ indices :

$$\begin{aligned} \mathcal{U}_{1I} &\equiv (\mathcal{U}^{1I})^* = \mathbb{C}_{I\dot{I}} \mathcal{U}^{2\dot{I}} \\ \mathcal{U}_{2I} &\equiv (\mathcal{U}^{2I})^* = -\mathbb{C}_{\dot{I}I} \mathcal{U}^{1\dot{I}} \,, \end{aligned} \tag{6.113}$$

where we have used the decomposition of the symplectic metric $\mathbb{C}_{\alpha\beta} = \begin{pmatrix} 0 & \mathbb{C}_{Ij} \\ \mathbb{C}_{jI} & 0 \end{pmatrix}$ with $\mathbb{C}_{Ij} = -\mathbb{C}_{jI} = \delta_{Ij}$.

JHEP03(2002)025

From equation (6.113) one finds that it is sufficient to refer to the $2n_H$ complex vielbein $\mathcal{U}^{1I}$,$\mathcal{U}^{2I}$ since the ones with dotted indices are related to them by complex conjugation.

Let us first examine the torsion-free equation obeyed by the quaternionic vielbein written in the decomposed indices:

$$d\mathcal{U}^{1I} + \frac{i}{2}\omega^3 \wedge \mathcal{U}^{1I} + \frac{i}{2}(\omega^1 - \mathrm{i}\omega^2) \wedge \mathcal{U}^{2I} + \Delta^I{}_J \wedge \mathcal{U}^{1J} + \Delta^I{}_{\dot J} \wedge \mathcal{U}^{1\dot J} = 0 \qquad (6.114)$$

$$d\mathcal{U}^{2I} - \frac{i}{2}(\omega^1 + \mathrm{i}\omega^2) \wedge \mathcal{U}^{1I} - \frac{i}{2}\omega^3 \wedge \mathcal{U}^{2I} + \Delta^I{}_J \wedge \mathcal{U}^{2J} + \Delta^I{}_{\dot J} \wedge \mathcal{U}^{2\dot J} = 0\,. \qquad (6.115)$$

For the $N = 1$ reduced Kähler-Hodge scalar manifold, the holonomy has to be $\mathrm{U}(1) \times \mathrm{SU}(n_H)$, with a non trivial U(1)-bundle, whose field-strength has to be identified with the Kähler form. Since in the $N = 2$ quaternionic parent theory there is a similar non trivial SU(2)-bundle, whose field-strength has to be identified with the Hyper-Kähler form, we assume that the U(1) part of the holonomy should be valued in the U(1) subgroup of the SU(2) valued connection of $N = 2$ quaternionic holonomy group.

From equations (6.114), (6.115) we see that, setting

$$\omega^1 = \omega^2 = \Delta^I_{\dot J} = 0 \qquad (6.116)$$

we get two Kähler-Hodge manifolds whose respective vielbeins obey the torsionless equations for each submanifold.

Let us now check the involution property dictated by the Frobenius theorem. As we know from section 3, this amounts to demand that the curvatures of the connections set to zero, eq. (6.116), must satisfy the constraints of being also zero on the submanifold. That is we must have:

$$\Omega^1 = \Omega^2 = \mathbb{R}^I{}_{\dot J} = 0\,, \qquad (6.117)$$

where the SU(2) curvature Ω^x is given by[10]

$$\Omega^x \equiv d\omega^x + \frac{1}{2}\epsilon^{xyz}\omega^y \wedge \omega^z = \mathrm{i}\,\lambda \mathbb{C}_{\alpha\beta}(\sigma^x)_{AB}\mathcal{U}^{\alpha A} \wedge \mathcal{U}^{\beta B} \qquad (6.119)$$

while the $\mathrm{Sp}(2n_H)$ curvature $\mathbb{R}^\alpha{}_\beta$ is given by:

$$\begin{aligned} \mathbb{R}^\alpha{}_\beta &\equiv d\Delta^\alpha{}_\beta + \Delta^\alpha{}_\gamma \wedge \Delta^\gamma{}_\beta \\ &= \lambda\epsilon_{AB}\mathcal{U}^{A\alpha} \wedge \mathcal{U}^B_\beta + \mathcal{U}^{A\gamma} \wedge \mathcal{U}^{B\delta}\epsilon_{AB}\mathbb{C}^{\alpha\rho}\Omega_{\rho\beta\gamma\delta}\,, \end{aligned} \qquad (6.120)$$

where $\Omega_{\alpha\beta\gamma\delta}$ is a completely symmetric 4-index tensor [2].

From equation (6.119) we see that the constraint (6.117) for involution is satisfied iff

$$\mathcal{U}^{1I} \wedge \mathcal{U}^{2I} = 0 \qquad (6.121)$$

that is if, say, the subset $\mathcal{U}^{2I} = \left(\mathcal{U}^{1\dot I}\right)^*$ of the quaternionic vielbein is set to zero on a submanifold $\mathcal{M}^{KH} \subset \mathcal{M}^Q$.

[10] Note that $\Omega^x = \lambda K^x_{uv}$ with K^x_{uv} given in terms of the three complex structures by:

$$\begin{aligned} K^x &= K^x_{uv} dq^u \wedge dq^v \\ K^x_{uv} &= h_{uw}(J^x)^w_v\,. \end{aligned} \qquad (6.118)$$

The scale λ is fixed by supersymmetry of the lagrangian and in our conventions is $\lambda = -1$.

JHEP03(2002)025

When condition (6.121) is imposed, our submanifold has dimension at most half the dimension of the quaternionic manifold (in the following we always refer to the maximal case, where $I = 1,\dots,n_H$) and the SU(2) connection is reduced to a $U(1)$ connection, whose curvature on $\mathcal{M}^{KH}$ is:

$$\Omega^3|_{\mathcal{M}^{KH}} = \mathrm{i}\lambda\mathcal{U}^{1I}\wedge\mathcal{U}_{1I} = \mathrm{i}\lambda\mathcal{U}^{1I}\wedge\overline{\mathcal{U}^{1I}} \tag{6.122}$$

so that the SU(2)-bundle of the quaternionic manifold is reduced to a U(1)-Hodge bundle for the n_H dimensional complex submanifold spanned by the n_H complex vielbein $\mathcal{U}^{1I}$.

The truncation corresponds therefore to select a n_H-complex dimensional submanifold $\mathcal{M}^{KH}\subset\mathcal{M}^Q$ spanned by the vielbein $\mathcal{U}^{1I}$ and to ask that, on the submanifold, the $2n_H$ extra degrees of freedom are frozen, that is:

$$\mathcal{U}^{2I}|_{\mathcal{M}^{KH}} = \left(\mathcal{U}_{1\dot I}\right)^*|_{\mathcal{M}^{KH}} = 0\,. \tag{6.123}$$

Calling w^s ($s = 1,\dots n_H$) a set of n_H holomorphic coordinates on $\mathcal{M}^{KH}$ and n^t ($t = 2n_H+1,\dots,4n_H$) a set of $2n_H$ real coordinates for the space orthogonal to $\mathcal{M}^{KH}$, we see that equation (6.123), which can be rewritten as:

$$\mathcal{U}^{2I}|_{\mathcal{M}^{KH}} = \left(\mathcal{U}^{2I}_s dw^s + \mathcal{U}^{2I}_{\overline{s}}d\overline{w}^{\overline{s}} + \mathcal{U}^{2I}_t dn^t\right)|_{\mathcal{M}^{KH}} = 0\,, \tag{6.124}$$

implies:

$$\mathcal{U}^{2I}_s|_{\mathcal{M}^{KH}} = \mathcal{U}^{2I}_{\overline{s}}|_{\mathcal{M}^{KH}} = 0 \tag{6.125}$$

since:

$$dn^t|_{\mathcal{M}^{KH}} = 0\,. \tag{6.126}$$

On the other hand, we also have:

$$\mathcal{U}^{1I}_t|_{\mathcal{M}^{KH}} = 0 \tag{6.127}$$

since the vielbein $\mathcal{U}^{1I}$ is tangent to the submanifold.

Let us note that the conditions (6.117) on the curvatures Ω^1,Ω^2 imposed on the submanifold do not imply that all their components are also zero there, and indeed from (6.125), (6.127) and the definition (6.119) it follows:

$$\Omega^1_{s\overline{s}}|_{\mathcal{M}^{KH}} = \Omega^1_{tt'}|_{\mathcal{M}^{KH}} = \Omega^2_{s\overline{s}}|_{\mathcal{M}^{KH}} = \Omega^2_{tt'}|_{\mathcal{M}^{KH}} = 0 \tag{6.128}$$

while the mixed components $\Omega^1_{st}|_{\mathcal{M}^{KH}}$, $\Omega^2_{st}|_{\mathcal{M}^{KH}}$ (together with their complex conjugates $\Omega^1_{\overline{s}t}|_{\mathcal{M}^{KH}}$, $\Omega^2_{\overline{s}t}|_{\mathcal{M}^{KH}}$) are different from zero. We also observe that, when the truncation is performed, also the mixed components of the metric are zero:

$$h_{st}|_{\mathcal{M}^{KH}} = \left(\mathcal{U}^{1I}_s\overline{\mathcal{U}}_{1I|t}\right)|_{\mathcal{M}^{KH}} = 0\,. \tag{6.129}$$

From (6.125), (6.127) and (6.119) it also follows that the only components different from zero of the 2-form Ω^3 are $\Omega^3_{s\overline{s}}$ and $\Omega^3_{tt'}$.

Let us now analyze in detail whether the involution of the constraint $\Delta^I{}_{\dot j} = 0$ is satisfied:

$$\begin{aligned}\mathbb{R}^I{}_{\dot j} &= \lambda\mathbb{C}_{\dot j K}\left(\mathcal{U}^{1I}\wedge\mathcal{U}^{2K} - \mathcal{U}^{2I}\wedge\mathcal{U}^{1K}\right) + \\ &\quad +2\mathbb{C}^{I\dot I}\Big[\mathcal{U}^{1K}\wedge\mathcal{U}^{2L}\Omega_{\dot I\dot J KL} + 2\mathcal{U}^{1K}\wedge\mathcal{U}^{2\dot L}\Omega_{\dot I\dot J K\dot L} + \mathcal{U}^{1\dot K}\wedge\mathcal{U}^{2\dot L}\Omega_{\dot I\dot J\dot K\dot L}\Big] = 0\,.\end{aligned} \tag{6.130}$$

JHEP03(2002)025

After imposing (6.123), the first line in eq. (6.130) is automatically zero, and eq. (6.130) is reduced to the constraint:

$$4\mathbb{C}^{I\dot{I}}\mathcal{U}^{1K}\wedge\mathcal{U}^{2\dot{L}}\Omega_{\dot{I}\dot{J}K\dot{L}}=0\,. \tag{6.131}$$

Furthermore, let us note that when the constraint (6.130) is imposed, the $\mathrm{Sp}(2n_H)$ holonomy gets reduced to $\mathrm{U}(1)\times\mathrm{SU}(n_H)$. The $\mathrm{U}(n_H)$ curvature becomes:

$$\mathbb{R}^{I\dot{J}}=\lambda\mathcal{U}^{1I}\wedge\mathcal{U}^{2\dot{K}}+\mathbb{C}^{I\dot{K}}\mathbb{C}^{\dot{J}L}\mathcal{U}^{1M}\wedge\mathcal{U}^{2\dot{N}}\Omega_{\dot{N}\dot{K}ML}\,. \tag{6.132}$$

Choosing cordinates such that $q^{4s+3}=q^{4s+4}=0, s=0,\dots,n_H-1$ we may introduce complex coordinates $w^s=q^{1+4s}+\mathrm{i}q^{2+4s}$ with Kähler 2-form $K=\Omega^3$ which is automatically closed. The $\mathrm{U}(1)$ connection of the Hodge bundle is given by ω^3_s as can be ascertained from the reduced form of eq. (6.114) expressing the vanishing of the torsion on the Kähler-Hodge submanifold $\mathcal{M}^{KH}$:

$$\nabla\mathcal{U}^{1I}\equiv d\mathcal{U}^{1I}+\frac{i}{2}\omega^3\wedge\mathcal{U}^{1I}+\Delta^I_J\wedge\mathcal{U}^{1J}=0\,. \tag{6.133}$$

In conclusion, what we have found is that the conditions for the truncation of a quaternionic manifold (spanning the scalar sector of n_H $N=2$ hypermultiplets) to a Kähler–Hodge one (spanning the scalar sector of n_h $N=1$ chiral multiplets) are the following:

- $\mathcal{U}^{2I}=\omega^1=\omega^2=\Delta^I_{\dot{J}}=0$
- The quaternionic manifold cannot be generic; in particular, the completely symmetric tensor $\Omega_{\alpha\beta\gamma\delta}\in\mathrm{Sp}(2n_H)$, appearing in the $\mathrm{Sp}(2n_H)$ curvature, must have the following constraint on its components:

$$\Omega_{\dot{I}\dot{J}K\dot{L}}=0\,. \tag{6.134}$$

The resulting submanifold, denoted by $\mathcal{M}^{KH}$, has at most n_H complex dimensions [10] and is of Kähler-Hodge type, with kählerian vielbein $P^I,\overline{P}^{\overline{I}}$ (for its normalization, see eq. (6.146) below):

$$\begin{aligned}\mathcal{U}^{1I}_u dq^u &\longrightarrow \frac{1}{\sqrt{2}}\overline{P}^{\overline{I}}_{\overline{s}}d\overline{w}^{\overline{s}}\\ \mathcal{U}^{2\dot{I}}_u dq^u &\longrightarrow \frac{1}{\sqrt{2}}P^I_s dw^s\end{aligned} \tag{6.135}$$

(where $w^s, s=1,\dots n_H$ are complex coordinates on the reduced Kähler manifold) and $\mathrm{U}(1)\times\mathrm{U}(n_H)$ curvature given by:

$$\mathcal{R}^I_J\equiv\mathcal{R}^u_v\mathcal{U}^{1I}_u\mathcal{U}^v_{1J}\longrightarrow\frac{\mathrm{i}}{2}\Omega^3\delta^I_J+\mathbb{R}^I{}_J\,. \tag{6.136}$$

Once the dimension of the manifold has been truncated, the only constraint on the quaternionic manifold is given by eq. (6.134). Let us therefore discuss how general it is, and which quaternionic manifolds satisfy it.

JHEP03(2002)025

First of all we note that the family of symmetric spaces $\mathrm{Sp}(2m,2)/\mathrm{Sp}(2)\times\mathrm{Sp}(2m)$ has a vanishing Ω-tensor, $\Omega_{\alpha\beta\gamma\delta}=0$ [11], and hence *a fortiori* satisfies our requirement.

Furthermore we can now show that the special quaternionic manifolds obtained by c-map [52] from special-Kähler manifolds do indeed satisfy the condition:

$$\Omega_{\dot{I}J K\dot{L}}=0\,. \tag{6.137}$$

Indeed the tensor (6.137) appears in eq. (6.130) multiplied by the product of the vielbein $\mathcal{U}^{1K}\wedge\mathcal{U}^{2\dot{L}}$. The same sub-block of the $\mathrm{Sp}(2n)$ curvature is denoted in [52] by $r'^{A}_{\overline{B}}$. Now, it is easy to recognise that the set of $n+1$ complex vielbein (v,e^a) of [52] have to be identified with our vielbein $\mathcal{U}^{1K}$. However, no wedge product of type $\overline{v}\wedge e^a$ nor of type $\overline{e}^a\wedge e^b$ appear in $r'^{A}_{\overline{B}}$, which means that the corresponding coefficient $\Omega_{\dot{I}JK\dot{L}}=0$. Therefore, all the special quaternionic manifolds (including non symmetric quaternionic spaces) can be reduced to Kähler-Hodge manifolds in a way consistent with our procedure.

There are however other symmetric spaces which do not correspond to c-map of special-Kähler manifolds, yet they satisfy our constraints. Indeed consider the following reduction from quaternionic to Kähler-Hodge manifolds:

$$\frac{\mathrm{SO}(4,n)}{\mathrm{SO}(4)\times\mathrm{SO}(n)}\longrightarrow\frac{\mathrm{SO}(2,n_1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_1)}\times\frac{\mathrm{SO}(2,n_2)}{\mathrm{SO}(2)\times\mathrm{SO}(n_2)}\,, \tag{6.138}$$

where $(n_1+n_2=n)$. We see that they satisfy our constraints. Indeed, the Kähler-Hodge manifold on the right of the correspondence in eq. (6.138) is apparently a submanifold of the corresponding quaternionic with half dimension. Therefore the conditions for the validity of the Frobenius theorem have to be satisfied, in particular eq. (6.134). Indeed, for symmetric spaces we can compute explicitly the Ω-tensor by comparing the general formula of the Riemann tensor for symmetric spaces:

$$\mathcal{R}^{uv}{}_{ts}\mathcal{U}^{\alpha A}_{u}\mathcal{U}^{\beta B}_{v}=-\frac{1}{2}f^{\alpha A|\beta B}{}_{h}f_{\gamma C|\delta D}{}^{h}\mathcal{U}^{\gamma C}_{[t}\mathcal{U}^{\delta D}_{s]}\,, \tag{6.139}$$

(where we have denoted by $f^{\alpha A|\beta B}{}_{\gamma C}$ the structure constants of the isometry group of the symmetric manifold $K=G/H$, the index h running on the Lie algebra of H, the couple of indices $A\alpha$ labelling the coset generators) with its general form in the case of quaternionic manifolds:

$$\mathcal{R}^{uv}{}_{ts}\mathcal{U}^{\alpha A}_{u}\mathcal{U}^{\beta B}_{v}=-\frac{\mathrm{i}}{2}\Omega^{x}_{ts}(\sigma_x)^{AB}C^{\alpha\beta}+\mathbb{R}^{\alpha\beta}_{ts}\,\epsilon^{AB}\,. \tag{6.140}$$

One easily obtains

$$\Omega_{\alpha\beta\gamma\delta}=-\frac{\lambda}{2}\left(\mathbb{C}_{\alpha\gamma}\mathbb{C}_{\beta\delta}+\mathbb{C}_{\alpha\delta}\mathbb{C}_{\beta\gamma}\right)-\frac{\mathrm{i}}{4}\epsilon^{AC}\epsilon^{BD}f_{C\{\alpha|\beta\}D|h}f^{h}{}_{A\{\gamma|\delta\}B}\,, \tag{6.141}$$

where the curly brackets mean symmetrization of the corresponding indices.

Using equation (6.141), we have explicitly verified the validity of (6.134) in the case of the omega-tensor appearing in (6.138). These quaternionic reductions explicitly appear in some effective lagrangians coming from superstring theory models [53].

JHEP03(2002)025

We still have to analyze the effects of the reduction on the hyperini and on the supersymmetry transformation laws. They become, after putting $\epsilon_2 = 0$:

$$\mathcal{U}^{1I}_u \delta q^u = \overline{\zeta}^I \epsilon^1 \tag{6.142}$$

$$\mathcal{U}^{2I}_u \delta q^u = -\mathbb{C}^{I\dot{J}} \overline{\zeta}_{\dot{J}} \epsilon_1 \tag{6.143}$$

$$\delta \zeta_I = \mathrm{i} \mathcal{U}^{2\dot{J}}_u \mathbb{C}_{I\dot{J}} \nabla_\mu q^u \gamma^\mu \epsilon^1 + g N^1_I \epsilon_1 = \left(\delta \zeta^I\right)^* \tag{6.144}$$

$$\delta \zeta_{\dot{J}} = \mathrm{i} \mathcal{U}^{2I}_u \mathbb{C}_{\dot{J}I} \nabla_\mu q^u \gamma^\mu \epsilon^1 + g N^1_{\dot{J}} \epsilon_1 = \left(\delta \zeta^{\dot{J}}\right)^* . \tag{6.145}$$

Choosing the normalization in such a way to match the normalization of the kinetic terms of the $N = 1$ theory, we set:

$$\mathcal{U}^{2\dot{J}} C_{I\dot{J}} = \frac{1}{\sqrt{2}} P_{Is} \tag{6.146}$$

$$N^1_I = \frac{1}{\sqrt{2}} P_{Is} N^s \tag{6.147}$$

$$\zeta^s \equiv \sqrt{2} P^{sI} \zeta_I = \sqrt{2} g^{s\bar{s}} \overline{P}^I_{\bar{s}} \zeta_I , \tag{6.148}$$

$$\mathcal{U}^{2\dot{J}}_u \mathbb{C}_{I\dot{J}} \nabla_\mu q^u |_{\mathcal{M}^{KH}} = \frac{1}{\sqrt{2}} P_{Is} \nabla_\mu w^s \tag{6.149}$$

which implies:

$$\mathcal{U}^{2\dot{J}}_u \mathbb{C}_{I\dot{J}} \delta q^u |_{\mathcal{M}^{KH}} = \frac{1}{\sqrt{2}} \overline{P}_{Is} \delta w^s , \tag{6.150}$$

where ζ^s denote chiral left-handed spinors with holomorphic world indices, P_{Is} are the vielbein of the Kähler-Hodge manifold $\mathcal{M}^{KH}$ and w^s its holomorphic coordinates. We observe that due to the definition (6.146) the 2-form Ω^3 defined in equation (6.122) is one half the Kähler 2-form on $\mathcal{M}^{KH}$.

In that way we obtain the standard formulae for the $N = 1$ supersymmetry transformation laws of the chiral multiplets (ζ^s, w^s), that is:

$$\begin{aligned} \delta\zeta^s &= \mathrm{i}\nabla_\mu w^s \gamma^\mu \epsilon^\bullet + N^s \epsilon_\bullet \\ \delta w^s &= \zeta^s \epsilon_\bullet , \end{aligned} \tag{6.151}$$

where

$$N^s \equiv \sqrt{2}\, g_{(\mathbf{\Lambda})} P^{sJ} N^1_J = 2\sqrt{2}\, g_{(\mathbf{\Lambda})} P^{sJ} \mathbb{C}_{J\dot{J}} \mathcal{U}^{1\dot{J}}_t k^t_{\mathbf{\Lambda}} \overline{L}^{\mathbf{\Lambda}} . \tag{6.152}$$

Note that the shift term N^s is indeed different from zero, but depends only on the isometries of the projected out part of the quaternionic manifold.[11] The explicit $N = 1$ form of the gauging contribution will be given in the next section 6.4.

From equation (6.143), however, we see that the condition $\mathcal{U}^{2I} = 0$ implies that the subset of n_H hyperinos $\zeta_{\dot{I}}$ have to be truncated out.

Consistency of the truncation in equation (6.145) implies

$$N^1_{\dot{J}} \equiv 2 g_{(\mathbf{\Lambda})} \mathbb{C}_{I\dot{J}} \mathcal{U}^{1I}_u k^u_{\mathbf{\Lambda}} \overline{L}^{\mathbf{\Lambda}} = 2 g_{(\mathbf{\Lambda})} \mathbb{C}_{I\dot{J}} \mathcal{U}^{1I}_s k^s_{\mathbf{\Lambda}} \overline{L}^{\mathbf{\Lambda}} = 0 \quad \Rightarrow \quad g_{(\mathbf{\Lambda})} k^s_{\mathbf{\Lambda}} \overline{L}^{\mathbf{\Lambda}} = 0 . \tag{6.153}$$

The restrictions on the theory imposed by this constraint will be discussed in the subsection 6.4.

[11] Indeed, the request $\mathcal{U}^{1\dot{I}}|_{\mathcal{M}} = \left(\mathcal{U}^{1\dot{I}}_s dw^s + \text{h.c.} + \mathcal{U}^{1\dot{I}}_t dn^t\right)|_{\mathcal{M}} = 0$ implies $\mathcal{U}^{1\dot{I}}_s = 0$ but does not impose any restriction on the components orthogonal to the retained submanifold.

JHEP03(2002)025

6.4 Further consequences of the gauging

The truncation $N=2 \to N=1$ implies, as we have seen in the previous subsections, a number of consequences that we are now going to discuss, and in particular:

- The D-term of the $N=1$-reduced gaugino $\lambda^\Lambda = -2 f_i^\Lambda \lambda^{i2}$ is:
$$D^\Lambda = W^{i21} = -2 g_{(\Lambda)} (\mathrm{Im} f)^{-1\Lambda\Sigma} \left(P^3_\Sigma(w^s) + P^0_\Sigma(z^i) \right). \tag{6.154}$$

- The $N=1$-reduced superpotential, that is the gravitino mass, is:
$$L(z,w) = \frac{\mathrm{i}}{2} g_{(X)} L^X \left(P^1_X - \mathrm{i} P^2_X \right). \tag{6.155}$$

- The fermion shifts of the $N=1$ chiral spinors $\chi^i = \lambda^{i1}$ coming from the $N=2$ gaugini are:
$$N^i = 2 g^{i\bar{\jmath}} \nabla_{\bar{\jmath}} \overline{L}. \tag{6.156}$$

- The fermion shifts of the $N=1$ chiral spinors ζ^s coming from $N=2$ hypermultiplets are:
$$N^s = -4 g_{(X)} k^t_X \overline{L}^X \mathcal{U}^{1\dot{I}}_t \mathcal{U}^s_{2\dot{I}}. \tag{6.157}$$

In order for the shifts given in eqs. (6.155), (6.156), (6.157) to define the correct transformation laws of the $N=1$ theory, we still have to show that the superpotential L is covariantly holomorphic with respect to the w^s coordinates:
$$\nabla_{\bar{s}} L = 0 \tag{6.158}$$
and that the N^s shift for the chiral multiplets coming from the quaternionic sector can be written with the standard expression for an $N=1$ chiral multiplets shift, that is as:
$$N^s = 2\, g\, g^{s\bar{s}} \nabla_{\bar{s}} \overline{L}. \tag{6.159}$$
These features do indeed follow, as a consequence of the reduction $\mathrm{SU}(2) \to \mathrm{U}(1)$ in the holonomy group. Indeed:
$$\nabla_{\bar{s}} L = \frac{\mathrm{i}}{2} L^{\mathbf{\Lambda}} \nabla_{\bar{s}} P^x_{\mathbf{\Lambda}} \left(\sigma^x\right)_1{}^2 = \mathrm{i} k^t_X L^X \Omega^x_{\bar{s}t} \left(\sigma^x\right)_1{}^2. \tag{6.160}$$
Now, recalling that:
$$\Omega^x \left(\sigma^x\right)_1{}^2 = 2\mathcal{U}^\alpha_1 \wedge \mathcal{U}^{2\beta} \mathbb{C}_{\alpha\beta} = 4\mathcal{U}^I_1 \wedge \mathcal{U}^{2\dot{J}} \mathbb{C}_{I\dot{J}} = 4\mathcal{U}^I_{1t} \mathcal{U}^{2\dot{J}}_s \mathbb{C}_{I\dot{J}} dn^t \wedge dw^s \tag{6.161}$$
we immediatly get: $\Omega^x_{st} \left(\sigma^x\right)_1{}^2 \neq 0$ while $\Omega^x_{\bar{s}t} \left(\sigma^x\right)_1{}^2 = 0$, so that $\nabla_{\bar{s}} L = 0$ follows.

Let us now compute N^s explicitly:
$$\begin{aligned} N^s &= \sqrt{2} P^{sJ} g N^1_J = 4 g_{(\mathbf{\Lambda})} \mathbb{C}_{J\dot{J}} g^{s\bar{s}} \mathcal{U}^J_{2\bar{s}} \mathcal{U}^{1\dot{J}}_t k^t_{\mathbf{\Lambda}} \overline{L}^{\mathbf{\Lambda}} \\ &= 4\, g_{(\mathbf{\Lambda})} \mathbb{C}_{J\dot{J}} g^{s\bar{s}} \frac{\mathrm{i}}{2} \Omega^x_{t\bar{s}} (\sigma^x)^1{}_2 k^t_X \overline{L}^X \\ &= -\mathrm{i}\, g_{(\mathbf{\Lambda})} g^{s\bar{s}} \nabla_{\bar{s}} P^x_X (\sigma^x)^1{}_2 \overline{L}^X = -\mathrm{i}\, g_{(\mathbf{\Lambda})} g^{s\bar{s}} \nabla_{\bar{s}} \left(P^1_X + \mathrm{i} P^2_X \right) \overline{L}^X \\ &= 2\, g_{(\mathbf{\Lambda})} g^{s\bar{s}} \nabla_{\bar{s}} \overline{L}, \end{aligned} \tag{6.162}$$
that is it has the right expression for an $N=1$ chiral shift, according to eq. (6.30).

JHEP03(2002)025

Let us now discuss the implications of the gauging constraints (6.37), (6.38) and (6.153) on the $N=1$ theory obtained by reduction, that is the consistency of the truncation of the second gravitino multiplet $\delta\psi_{\mu 2}=0$ and of the spinors $\zeta_{\dot I}$ in the hypermultiplets sector for the gauged theory:

$$\begin{aligned}
\widehat{\omega}_1{}^2 &= 0 \Longrightarrow g_{(\mathbf{\Lambda})}A^{\mathbf{\Lambda}}\left(P^1_{\mathbf{\Lambda}}-\mathrm{i}P^2_{\mathbf{\Lambda}}\right)=0 && (6.163)\\
S_{12} &= 0 \Longrightarrow g_{(\mathbf{\Lambda})}L^{\mathbf{\Lambda}}P^3_{\mathbf{\Lambda}}=0 && (6.164)\\
\delta\zeta_{\dot I} &= 0 \Longrightarrow g_{(\mathbf{\Lambda})}k^s_{\mathbf{\Lambda}}\overline{L}^{\mathbf{\Lambda}}=0\,. && (6.165)
\end{aligned}$$

Since the vectors of the $N=2$ theory which are not gauged do not enter in the previous equations we may limit ourselves to consider the case where the index $\mathbf{\Lambda}$ runs over the adjoint representation of the $N=2$ gauge group. If we call $G^{(2)}$ the gauge group of the $N=2$ theory and $G^{(1)}\subseteq G^{(2)}$ the gauge group of the corresponding $N=1$ theory, then we have that the adjoint representation of $G^{(2)}$ decomposes as

$$Adj(G^{(2)}) \Rightarrow Adj(G^{(1)})+R(G^{(1)})\,, \qquad (6.166)$$

where $R(G^{(1)})$ denotes some representation of $(G^{(1)})$ (the representation $R(G^{(1)})$ is of course absent for $G^{(1)}=G^{(2)}$). The gauged vectors of the $N=1$ theory are restricted to the subset $\{A^\Lambda\}$ generating $Adj(G^{(1)})$ (that is the index $\mathbf{\Lambda}$ is decomposed as $\mathbf{\Lambda}\to(\Lambda,X)$, with $\Lambda\in Adj(G^{(2)}$ and $X\in R(G^{(1)})$).

This decomposition of the indices is of course the same as the one used in analyzing the consequences of the constraint (6.34) in section 6.2. In particular, the graviphoton index $\mathbf{\Lambda}=0$ always belongs to the set X since the graviphoton A^0 is projected out.

The quaternionic Killing vectors of the $N=2$ theory then decompose as

$$k^u_{\mathbf{\Lambda}} \Rightarrow \{k^s_\Lambda, k^{\overline{s}}_\Lambda, k^t_\Lambda, k^s_X, k^{\overline{s}}_X, k^t_X\}\,. \qquad (6.167)$$

Obviously, we must have that $k^s_X=0$ since the Killing vectors of the reduced submanifold have to span the adjoint representation of $G^{(1)}$. Viceversa, the Killing vectors with world index in the orthogonal complement, $k^t_{\mathbf{\Lambda}}$, must obey $k^t_\Lambda=0$, while k^t_X are in general different from zero. Indeed, the isometries generated by $k^t_{\mathbf{\Lambda}}$ would not leave invariant the hypersurface describing the submanifold $\mathcal{M}^{KH}\subset\mathcal{M}^Q$. These properties will be in fact confirmed in appendix E, by a careful analysis of the reduction of the quaternionic Ward identities.

Coming back to the implications of the constraints (6.163), (6.164), (6.165), they can be rewritten, using the results of section 6.2, as follows:

$$\begin{aligned}
g_{(\Lambda)}A^\Lambda\left(P^1_\Lambda-\mathrm{i}P^2_\Lambda\right) &= 0 && (6.168)\\
g_{(X)}L^X P^3_X &= 0 && (6.169)\\
g_{(X)}k^s_X\overline{L}^X &= 0\,. && (6.170)
\end{aligned}$$

Since we have found that $k^s_X=0$, eq. (6.170) is identically satisfied.

Eqs. (6.168) and (6.169) are satisfied by requiring:

$$P^1_\Lambda=P^2_\Lambda=0\,;\quad P^3_X=0\,. \qquad (6.171)$$

JHEP03(2002)025

Then the superpotential of the theory is given by [12]–[22]:

$$L = \frac{\mathrm{i}}{2} L^X(z,\overline{z}) \left(P^1_X(w,\overline{w}) - \mathrm{i} P^2_X(w,\overline{w}) \right) . \tag{6.172}$$

We are left with an $N=1$ theory coupled to n'_V vector multiplets ($\Lambda = 1, \ldots, n'_V$) and $n_C + n_H$ chiral multiplets ($X = 0, 1, \ldots, n_C$) with superpotential (6.172). All the isometries of the scalar manifolds are in principle gauged since the D-term of the reduced $N=1$ theory depends on $P^0_\Lambda(z,\overline{z}) + P^3_\Lambda(w,\overline{w})$.

In the particular case where the gauge group $G^{(1)}$ of the $N=1$ reduced theory is the same as the gauge group $G^{(2)}$ of the $N=2$ parent theory, the index X takes only the value zero and all the scalars are truncated out ($L^\Lambda = 0, L^{\hat{0}} = 1$). The vectors A^Λ are retained in the truncation while A^0 is projected out. In this case the superpotential reduces to:

$$L = \frac{\mathrm{i}}{2} L^0 \left(P^1_0 - \mathrm{i} P^2_0 \right) . \tag{6.173}$$

Moreover, from eq. (6.93) we have that in this case the prepotential $P^0_\Lambda = 0$, and the D-term depends only on $P^3_\Lambda(w,\overline{w})$. We then have an $N=1$ theory coupled to n_V vector multiplets and n_H chiral multiplets, with gauged isometries and superpotential (6.173).

Note that when $P^1_0 - \mathrm{i} P^2_0$ is constant, (6.173) gives a constant F-term. This case can only be obtained in absence of hypermultiplets. Indeed, from the general quaternionic formula [54]

$$n_H P^x_{\mathbf{\Lambda}} = -\frac{1}{2} \Omega^x_{uv} \nabla^u k^v_{\mathbf{\Lambda}} \tag{6.174}$$

we see that if $n_H \neq 0$ a Fayet-Iliopoulos term, as well as a constant F-term, is excluded [54], since a constant P^x_0 is not compatible with the covariance of the r.h.s. under SU(2) and the gauge group. Even when the theory is ungauged ($k^u_{\mathbf{\Lambda}} = 0$) a constant P^x_0 is still excluded for $n_H \neq 0$, since in this case equation (6.174) reduces to $n_H P^x_{\mathbf{\Lambda}} = 0$, implying $P^x_{\mathbf{\Lambda}} = 0$.

If $n_H = 0$, then a constant $P^x_{\mathbf{\Lambda}}$ is possible ($N=2$ Fayet-Iliopoulos term) [55],[12] provided the gauge group is abelian (otherwise it breaks the gauge group) and provided it satisfies the identity

$$\epsilon^{xyz} P^y_{\mathbf{\Lambda}} P^z_{\mathbf{\Sigma}} = 0 \tag{6.175}$$

which follows from the general quaternionic Ward identity [5, 11]

$$\frac{1}{\lambda} \Omega^x_{uv} k^u_{\mathbf{\Lambda}} k^v_{\mathbf{\Sigma}} + \frac{1}{2} \epsilon^{xyz} P^y_{\mathbf{\Lambda}} P^z_{\mathbf{\Sigma}} - \frac{1}{2} f_{\mathbf{\Lambda\Sigma}}{}^{\mathbf{\Gamma}} P^x_{\mathbf{\Gamma}} = 0 \tag{6.176}$$

in absence of hypermultiplets.

When we reduce the theory to $N=1$, a constant value of $P^{\mathrm{i}}_X \equiv \xi^{\mathrm{i}}_X \neq 0$ ($\mathrm{i} = 1,2$) or $P^3_\Lambda \equiv \xi^3_\Lambda \neq 0$ are both compatible with all the constraints (6.168)–(6.170); in particular $L^{\mathbf{\Lambda}} \xi^3_{\mathbf{\Lambda}} = 0$ and $A^{\mathbf{\Lambda}} \xi^{\mathrm{i}}_{\mathbf{\Lambda}} = 0$ implying the presence of a $N=1$ Fayet-Iliopoulos term corresponding to ξ^3_Λ, or a constant F-term corresponding to ξ^{i}_X.

[12] An $N=2$ Fayet-Iliopoulos term coming from P^0_Λ is excluded by the Ward identity (6.92).

JHEP03(2002)025

6.5 $N=2 \to N=1$ scalar potential

Let us now compute explicitely the reduction of the scalar potential of the $N=2$ theory down to $N=1$. The $N=2$ scalar potential is given by:

$$\begin{aligned}\mathcal{V}^{N=2} &= \left(g_{\mathcal{I}\overline{\mathcal{J}}}k^{\mathcal{I}}_{\mathbf{\Lambda}}k^{\overline{\mathcal{J}}}_{\mathbf{\Sigma}} + 4h_{uv}k^u_{\mathbf{\Lambda}}k^v_{\mathbf{\Sigma}}\right)\overline{L}^{\mathbf{\Lambda}}L^{\mathbf{\Sigma}} + \left(-\frac{1}{2}(\mathrm{Im}\mathcal{N}^{-1})^{\mathbf{\Lambda\Sigma}} - \overline{L}^{\mathbf{\Lambda}}L^{\mathbf{\Sigma}}\right)P^x_{\mathbf{\Lambda}}P^x_{\mathbf{\Sigma}} - \\ &\quad -3P^x_{\mathbf{\Lambda}}P^x_{\mathbf{\Sigma}}\overline{L}^{\mathbf{\Lambda}}L^{\mathbf{\Sigma}}\end{aligned} \tag{6.177}$$

while the $N=1$ scalar potential can be written in terms of the covariantly holomorphic superpotential L as:

$$\mathcal{V}^{N=1} = 4\left(\nabla_\ell L\nabla_{\overline{\ell}}\overline{L}g^{\ell\overline{\ell}} - 3|L|^2 + \frac{1}{16}\mathrm{Im}f_{\Lambda\Sigma}D^\Lambda D^\Sigma\right), \tag{6.178}$$

where the holomorphic index ℓ runs over all the scalars of the theory.

Before performing the reduction it is instructive to work out in detail the supersymmetry Ward identity involving the scalar potential [56, 57]:

$$\delta^A_B\mathcal{V}^{N=2} = -12\overline{S}^{AC}S_{CB} + g_{\mathcal{I}\overline{\mathcal{J}}}W^{\mathcal{I}AC}W^{\overline{\mathcal{J}}}_{BC} + 2N^A_\alpha N^\alpha_B\,. \tag{6.179}$$

Instead of taking the trace of (6.179) on the SU(2) indices A,B, thus recovering the potential (6.177), one can alternatively write down the stronger relations:

$$\delta^1_1\mathcal{V}^{N=2} = \mathcal{V}^{N=2} = -12\overline{S}^{1C}S_{C1} + g_{\mathcal{I}\overline{\mathcal{J}}}W^{\mathcal{I}1C}W^{\overline{\mathcal{J}}}_{1C} + 2N^1_\alpha N^\alpha_1 \tag{6.180}$$

$$\delta^2_2\mathcal{V}^{N=2} = \mathcal{V}^{N=2} = -12\overline{S}^{2C}S_{2C} + g_{\mathcal{I}\overline{\mathcal{J}}}W^{\mathcal{I}2C}W^{\overline{\mathcal{J}}}_{C2} + 2N^2_\alpha N^\alpha_2 \tag{6.181}$$

and furthermore:

$$\delta^2_1\mathcal{V}^{N=2} = 0 = -12\overline{S}^{2C}S_{1C} + g_{\mathcal{I}\overline{\mathcal{J}}}W^{iC2}W^{\overline{\mathcal{J}}}_{C1} + 2N^2_\alpha N^\alpha_1\,. \tag{6.182}$$

When we pass to the truncated theory, the matrix S_{AB} becomes diagonal ($S_{12} \sim P^3_\Lambda\overline{L}^\Lambda = 0$) and its eigenvalues are the masses of the 2 gravitini:

$$S_{AB} = \begin{pmatrix} L & 0 \\ 0 & \widetilde{L}\end{pmatrix}, \tag{6.183}$$

where:

$$L = \frac{\mathrm{i}}{2}L^X(P^1_X - \mathrm{i}P^2_X) \tag{6.184}$$

$$\widetilde{L} = \frac{\mathrm{i}}{2}L^X(-P^1_X - \mathrm{i}P^2_X) \tag{6.185}$$

so that:

$$\begin{aligned}|L|^2 &= S_{11}S^{11} = \frac{1}{4}L^X\overline{L}^Y\left[P^x_XP^x_Y + \mathrm{i}\left(P^1_XP^2_Y - P^2_XP^1_Y\right)\right] \\ |\widetilde{L}|^2 &= S_{22}S^{22} = \frac{1}{4}L^X\overline{L}^Y\left[P^x_XP^x_Y - \mathrm{i}\left(P^1_XP^2_Y - P^2_XP^1_Y\right)\right].\end{aligned} \tag{6.186}$$

JHEP03(2002)025

The difference between the 2 gravitino mass eigenvalues can be written in terms of the fermionic shifts as:

$$
\begin{aligned}
|L|^2 - |\widetilde{L}|^2 &= \frac{\mathrm{i}}{2}\left(P^1_X P^2_Y - P^2_X P^1_Y\right) = \overline{S}^{1C} S_{C1} - \overline{S}^{2C} S_{C2} \\
&= \frac{1}{12}\left(g_{i\bar{\jmath}} W^{i1C} W^{\bar{\jmath}}_{1C} + 2N^1_\alpha N^\alpha_1 - g_{i\bar{\jmath}} W^{i2C} W^{\bar{\jmath}}_{2C} - 2N^2_\alpha N^\alpha_2\right). \qquad (6.187)
\end{aligned}
$$

Let us now perform the reduction. Using, e.g., equation (6.180) and recalling that $S_{12} = 0$ and $N^1_I = 0$ (see eq. (6.153)), we find :

$$
\mathcal{V}^{N=2\to N=1} = -12\overline{S}^{11} S_{11} + g_{\mathcal{I}\overline{\mathcal{J}}}\left(W^{\mathcal{I}11} W^{\overline{\mathcal{J}}}_{11} + W^{\mathcal{I}12} W^{\overline{\mathcal{J}}}_{12}\right) + 2N^1_I N^I_1 \,. \qquad (6.188)
$$

Using equations (6.44), (6.43), the first two terms of equation (6.188) give:

$$
\begin{aligned}
-12\overline{S}^{11} S_{11} &= -3P^{\mathrm{i}}_X P^{\mathrm{i}}_Y L^X \overline{L}^Y + 3\mathrm{i}\left(P^2_X P^1_Y - P^1_X P^2_Y\right) L^X \overline{L}^Y = -12L\overline{L} \qquad (6.189) \\
g_{\mathcal{I}\overline{\mathcal{J}}} W^{\mathcal{I}11} W^{\overline{\mathcal{J}}}_{11} &= \left(P^1_X + \mathrm{i}P^2_X\right)\left(P^1_Y - \mathrm{i}P^2_Y\right) U^{XY} = 4g^{k\bar{l}} \nabla_k L \nabla_{\bar{l}} \overline{L} \,. \qquad (6.190)
\end{aligned}
$$

For the term $g_{\mathcal{I}\overline{\mathcal{J}}} W^{\mathcal{I}21} W^{\overline{\mathcal{J}}}_{21}$ we obtain:

$$
g_{\mathcal{I}\overline{\mathcal{J}}} W^{\mathcal{I}21} W^{\overline{\mathcal{J}}}_{21} = -2\mathrm{Im}\mathcal{N}_{\mathbf{\Lambda\Sigma}} f^{\mathbf{\Lambda}}_{\mathcal{I}} f^{\mathbf{\Sigma}}_{\overline{\mathcal{J}}} W^{\mathcal{I}21} W^{\overline{\mathcal{J}}}_{21} = -\frac{1}{2}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} D^\Lambda D^\Sigma = \frac{1}{4}\mathrm{Im} f_{\Lambda\Sigma} D^\Lambda D^\Sigma \,, \qquad (6.191)
$$

where we have reduced the indices according to the results of subsection 6.3 and used equations (6.98), (6.97), (6.99).

To compute the last term in eq. (6.188) we use eq. (6.147) and (6.159) and we find

$$
2N^1_I N^I_1 = g_{s\bar{s}} N^s \overline{N}^{\bar{s}} = 4g^{s\bar{s}} \nabla_s L \nabla_{\bar{s}} \overline{L} \,. \qquad (6.192)
$$

Collecting all the terms we find that the reduction of the $N = 2$ scalar potential gives:

$$
\mathcal{V}^{N=2\to N=1} = 4\left[-3L\overline{L} + g^{i\bar{\jmath}} \nabla_i L \nabla_{\bar{\jmath}} \overline{L} + g^{s\bar{s}} \nabla_s L \nabla_{\bar{s}} \overline{L} + \frac{1}{16}\mathrm{Im} f_{\Lambda\Sigma} D^\Lambda D^\Sigma\right] \qquad (6.193)
$$

which coincides with the scalar potential (6.178) of the $N = 1$ theory, where we have decomposed the indices according to the fact that the σ-model is a product manifold .

We note that our computation of the reduction of the scalar potential has been performed by first reducing the $N = 2$ fermionic shifts to $N = 1$ and then computing the potential. Of course, we could also have performed the computation by directly computing the reduction of each term of the $N = 2$ potential. In the latter case, to obtain the desired results requires some non trivial computations. In particular, there are some subtleties related to the observation that the $N = 2$ potential does not contain "interference" contributions of the form $P^0_{\mathbf{\Lambda}} P^x_{\mathbf{\Sigma}}$ or $P^x_{[\mathbf{\Lambda}} P^y_{\mathbf{\Sigma}]}$, while such terms are instead present in the $N = 1$ potential, given the form (6.154) of the D-term and (6.155) of the superpotential. To solve the puzzle and recover the precise correspondence between the $N = 2$ and $N = 1$ theories, one has to use several times the reduced forms of the Ward identities of quaternionic and special-Kähler geometries, the definition of the quaternionic Killing vectors [45, 54, 58] and the expression that the special geometry prepotential gets in the reduction, equation (6.93). The explicit computation is given in appendix F.

JHEP03(2002)025

$N=2$ $(n_V=0)$, $\mathcal{M}_Q$ $(\dim_Q = n)$	$N=1$ $(n_V=0)$, $\mathcal{M}_{KH}$ $(\dim_C = n)$
$\frac{\mathrm{U}(2,n+1)}{\mathrm{U}(2)\times\mathrm{U}(n+1)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{U}(1,n)}{\mathrm{U}(1)\times\mathrm{U}(n)}$
$\frac{\mathrm{SO}(4,n+1)}{\mathrm{SO}(4)\times\mathrm{SO}(n+1)}$ $(n\geq 2)$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(2,n-1)}{\mathrm{SO}(2)\times\mathrm{SO}(n-1)}$
$\frac{G_{2(2)}}{\mathrm{SO}(4)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$
$\frac{F_{4(4)}}{\mathrm{USp}(6)\times\mathrm{USp}(2)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{Sp}(6,\mathbb{R})}{\mathrm{U}(3)}$
$\frac{E_{6(2)}}{\mathrm{SU}(6)\times\mathrm{SU}(2)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3)\times\mathrm{SU}(3)\times\mathrm{U}(1)}$
$\frac{E_{7(-8)}}{\mathrm{SO}(12)\times\mathrm{SU}(2)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SO}^*(12)}{\mathrm{U}(6)}$
$\frac{E_{8(-24)}}{E_7\times\mathrm{SU}(2)}$	$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{E_{7(-26)}}{E_6\times\mathrm{SO}(2)}$

Table 5: $N=2\to N=1$.

6.6 Examples of truncation to $N=1$ gauged supergravity

As an application of the formalism developed in this section, we can now consider reduction on $N=8$ to $N=1$ or in general of $N=2$ theories down to $N=1$.

The simplest case is to consider $N=2$ special-Kähler manifolds which are also $N=1$ Hodge-Kähler, or submanifolds of half the dimension of quaternionic manifolds which are "dual" (under c-map) to special-Kähler.

We first consider "dual quaternionic manifolds" which are symmetric spaces; they were all given in [39, table 4]. This immediately gives the $N=2\to N=1$ reduction of theories with only hypermultiplets as follows:

It is interesting to note that if $\mathcal{M}_Q=\frac{G_Q}{H_Q}$, $\mathcal{M}_{SK}=\frac{\mathrm{SU}(1,1)\times G}{\mathrm{U}(1)\times H}$ then $H_Q=\mathrm{SU}(2)\times G_c$, where G_c is the compact form of G!.

From the previous table we can immediately obtain $N=1$ truncations of $N=8$ supergravity with (n_V,n_H) replaced by $(n_V^{(N=1)}, n_C=n_V+n_Q)$.

In all these models (unless $n_Q=0$) the Kähler-Hodge manifold will be of the form

$$SK(n_V)\times SK(n_Q-1)\times\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\,. \tag{6.194}$$

As a simple example, motivated by string construction [53], for the application of the results of the previous sections, we consider a $N=4$, $D=4$ matter coupled supergravity with gauge group $\mathrm{SO}(n)$ (n even). The σ-model of the scalars in presence of gauging is given by:

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\times\frac{\mathrm{SO}(6,\frac{n(n-1)}{2})}{\mathrm{SO}(6)\times\mathrm{SO}(\frac{n(n-1)}{2})}\,, \tag{6.195}$$

and the content of the scalar sector can be encoded in the vielbein 1-form P_{ABI} where the antisymmetric couple AB labels the irrep. $\mathbf{6}\in\mathrm{SU}(4)$ and I labels the fundamental representation of $\mathrm{SO}(\frac{n(n-1)}{2})$.

This $N=4$ theory is reduced to $N=2$ through the action of a $\mathbb{Z}_2$ group and to $N=1$ by the action of $\mathbb{Z}_2\times\mathbb{Z}_2'$. The generators of $\mathbb{Z}_2\times\mathbb{Z}_2'$ in the R-symmetry group SU(4) are

JHEP03(2002)025

given by:

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\pi} & 0 \\ 0 & 0 & 0 & e^{i\pi} \end{pmatrix} ; \qquad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\pi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\pi} \end{pmatrix} \tag{6.196}$$

so that two gravitinos are singlets with respect to $\mathbb{Z}_2$ and one gravitino is invariant with respect to $\mathbb{Z}_2 \times \mathbb{Z}'_2$.

To obtain charged matter in the $N = 4 \to N = 2$ reduction, we implement the action of $\mathbb{Z}_2$ on the gauge group. Let us make the following decomposition

$$\mathrm{SO}(n) \xrightarrow{\mathbb{Z}_2} \mathrm{SO}(n_A) \times \mathrm{SO}(n_B) \tag{6.197}$$

so that, under the action of $\mathbb{Z}_2$:

$$\begin{aligned} n_A &\Rightarrow n_A \\ n_B &\Rightarrow \alpha n_B \end{aligned} \tag{6.198}$$

and then

$$\begin{aligned} Adj(\mathrm{SO}(n_A)) &\xrightarrow{\mathbb{Z}_2} Adj(\mathrm{SO}(n_A)) \\ Adj(\mathrm{SO}(n_B)) &\xrightarrow{\mathbb{Z}_2} Adj(\mathrm{SO}(n_B)) \\ (n_A, n_B) &\xrightarrow{\mathbb{Z}_2} \alpha(n_A, n_B)\,. \end{aligned} \tag{6.199}$$

Correspondingly, for the group SU(4) we have:

$$\begin{aligned} \mathbf{4} &\xrightarrow{\mathbb{Z}_2} \alpha\mathbf{4} \\ \mathbf{2}_1 &\xrightarrow{\mathbb{Z}_2} \mathbf{2}_1\,. \end{aligned} \tag{6.200}$$

The scalars transforming non trivially under $\mathbb{Z}_2$ are projected out and we are left with the coset manifold:

$$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}\left(2, \frac{n_A(n_A-1)}{2} + \frac{n_B(n_B-1)}{2}\right)}{\mathrm{SO}(2) \times \mathrm{SO}\left(\frac{n_A(n_A-1)}{2} + \frac{n_B(n_B-1)}{2}\right)} \times \frac{\mathrm{SO}(4, n_A n_B)}{\mathrm{SO}(4) \times \mathrm{SO}(n_A n_B)}\,, \tag{6.201}$$

where the first two factors define an $N = 2$ special-Kähler manifold and the last factor is a quaternionic manifold.

In order to obtain an $N = 1$ supergravity theory, the gauge groups $\mathrm{SO}(n_A)$ and $\mathrm{SO}(n_B)$ are further decomposed as follows:

$$\begin{aligned} \mathrm{SO}(n_A) &\to \mathrm{SO}(n_1) \times \mathrm{SO}(n_2) \\ \mathrm{SO}(n_B) &\to \mathrm{SO}(n_3) \times \mathrm{SO}(n_4) \end{aligned} \tag{6.202}$$

and we define the action of $\mathbb{Z}'_2$ as:

$$\begin{aligned} n_1 &\Rightarrow n_1\,, \qquad n_2 \Rightarrow \beta n_2 \\ n_3 &\Rightarrow n_3\,, \qquad n_4 \Rightarrow \beta n_4\,. \end{aligned} \tag{6.203}$$

JHEP03(2002)025

This induces an action of $\mathbb{Z}_2 \times \mathbb{Z}'_2$ on the decomposition of the gauge group:

$$
\begin{aligned}
Adj(\mathrm{SO}(n)) &\xrightarrow{\mathbb{Z}_2} Adj(\mathrm{SO}(n_A)) + Adj(\mathrm{SO}(n_B)) + (n_A, n_B)_\alpha \\
&\xrightarrow{\mathbb{Z}_2\times\mathbb{Z}'_2} Adj(\mathrm{SO}(n_1))_1 + Adj(\mathrm{SO}(n_2))_1 + Adj(\mathrm{SO}(n_3))_1 + Adj(\mathrm{SO}(n_4))_1 + \\
&\quad +(n_1, n_2, 1, 1)_\beta + (1, 1, n_3, n_4)_\beta + (n_1, 1, n_3, 1)_\alpha + \\
&\quad +(n_1, 1, 1, n_4)_{\alpha\beta} + (1, n_2, n_3, 1)_{\alpha\beta} + (1, n_2, 1, n_4)_\alpha \,.
\end{aligned} \tag{6.204}
$$

In equation (6.204) we have labelled each representation with indices $1, \alpha, \beta, \alpha\beta$ whose meaning is that the corresponding representation is invariant or transforms under α, β or $\alpha\beta$ respectively. That is the representations $Adj(\mathrm{SO}(n_I))$ are invariant under $\mathbb{Z}_2 \times \mathbb{Z}'_2$, while the remaining bifundamental representations (n_I, n_J) transform as follows:

$$
\begin{aligned}
&(n_1, n_3); (n_2, n_4) \text{ transform under } \alpha \\
&(n_1, n_2); (n_3, n_4) \text{ transform under } \beta \\
&(n_2, n_3); (n_1, n_4) \text{ transform under } \alpha\beta \,.
\end{aligned} \tag{6.205}
$$

With the same notation, let us now consider the $\mathbb{Z}_2 \times \mathbb{Z}'_2$ action on the $\mathbf{6}$ of SU(4):

$$
\begin{aligned}
\mathbf{6} &\xrightarrow{\mathbb{Z}_2} 4_\alpha + 2_1 \\
&\xrightarrow{\mathbb{Z}_2\times\mathbb{Z}'_2} (2_\alpha + 2_{\alpha\beta}) + 2_\beta \,.
\end{aligned} \tag{6.206}
$$

Joining the information coming from the the decomposition of SU(4) and SO $(n(n-1)/2)$ we see that the scalars which remain invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}'_2$ are given by the vielbein in the following representations: $P_{2_\alpha(n_1,n_3)}$;$P_{2_\alpha(n_2,n_4)}$; $P_{2_\beta(n_1,n_2)}$;$P_{2_\beta(n_3,n_4)}$; $P_{2_{\alpha\beta}(n_1,n_4)}$;$P_{2_{\alpha\beta}(n_2,n_3)}$. This means that the special-Kähler manifold reduces to:

$$
\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}\left(2, \frac{n_A(n_A-1)}{2} + \frac{n_B(n_B-1)}{2}\right)}{\mathrm{SO}(2) \times \mathrm{SO}\left(\frac{n_A(n_A-1)}{2} + \frac{n_B(n_B-1)}{2}\right)} \to \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2, n_1n_2 + n_3n_4)}{\mathrm{SO}(2) \times \mathrm{SO}(n_1n_2 + n_3n_4)} \tag{6.207}
$$

while the quaternionic manifold splits as follows:

$$
\frac{\mathrm{SO}(4, n_An_B)}{\mathrm{SO}(4) \times \mathrm{SO}(n_An_B)} \to \frac{\mathrm{SO}(2, n_1n_3 + n_2n_4)}{\mathrm{SO}(2) \times \mathrm{SO}(n_1n_3 + n_2n_4)} \times \frac{\mathrm{SO}(2, n_1n_4 + n_2n_3)}{\mathrm{SO}(2) \times \mathrm{SO}(n_1n_4 + n_2n_3)} \,. \tag{6.208}
$$

Let us now comment this result.

From the analysis in section 6.2 we have learnt that when we reduce a gauged $N = 2$ theory to $N = 1$ (with $G^{(2)} \to G^{(1)}$) the surviving scalars from the vector multiplets sector are those which are in the representation $R(G^{(1)})$ according to eq. (6.166), while all the scalars in the adjoint representation of $G^{(1)}$ are truncated out. Precisely this happens in our case. Indeed, from equation (6.207) the irreps (n_1, n_2) and (n_3, n_4) belong to the left over representations in eq. (6.204). Furthermore, all the other bifundamental rep. belong to the scalars coming from the quaternionic sector, according to equation (6.208). Note that the total dimensional of the product manifold of equation (6.208) is exactly half

JHEP03(2002)025

the dimension of the parent quaternionic manifold, according to the general result found in section 6.1. It is interesting to observe that the same kind of result appears in the decomposition $N = 4 \to N = 2$ described by eq. (6.201). In fact, the reduced product manifold in eq. (6.201) has a σ-model whose scalars belong again to the representation $R = (n_A, n_B)$ left over in the reduction of the adjoint representation of the $N = 4$ gauge group.

Other examples can be obtained [41] from heterotic strings compactified on $\mathbb{Z}_N$ orbifolds with reduced non abelian gauge group E_6.

We finally observe that the $N = 2$ special-Kähler manifold in the l.h.s. of (6.207) can be parametrized with the symplectic section $(L^{\mathbf{\Lambda}}, M_{\mathbf{\Lambda}} = \eta_{\mathbf{\Lambda\Sigma}} S L^{\mathbf{\Sigma}})$ (with $L^{\mathbf{\Lambda}} L^{\mathbf{\Sigma}} \eta_{\mathbf{\Lambda\Sigma}} = 0$ and $\eta_{\mathbf{\Lambda\Sigma}} = (1, 1, -1, \dots, -1)$) where a prepotential F does not exist [59]. In this case the $N = 2$ vector kinetic matrix has the form:

$$\mathcal{N}_{\mathbf{\Lambda\Sigma}} = (S - \overline{S}) \left(\Phi_{\mathbf{\Lambda}} \overline{\Phi}_{\mathbf{\Sigma}} + \overline{\Phi}_{\mathbf{\Lambda}} \Phi_{\mathbf{\Sigma}}\right) + \overline{S} \eta_{\mathbf{\Lambda\Sigma}}\,; \quad \Phi_{\mathbf{\Lambda}} \equiv \frac{L^{\mathbf{\Lambda}}}{\sqrt{L^{\mathbf{\Lambda}} \overline{L}_{\mathbf{\Lambda}}}}\,. \tag{6.209}$$

When we perform the truncation to $N = 1$, the sections L^{Λ} become zero, and the $N = 1$ vector kinetic matrix takes the form:

$$\mathcal{N}_{\Lambda\Sigma} = \overline{S} \eta_{\Lambda\Sigma}\,, \tag{6.210}$$

that is it becomes antiholomorphic in the complex scalar S parametrizing the manifold $\mathrm{SU}(1,1)/\mathrm{U}(1)$.

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A. Supersymmetry reduction from superspace Bianchi identities

We want now to show that the constraints found at the level of supersymmetry transformation laws are actually sufficient to guarantee the closure of the supersymmetry algebra of the reduced theory.

We prove this statement by considering the reduction of the superspace Bianchi identities of the $N = 8$ theory (which, as is well known, is equivalent to the "on-shell" closure of the supersymmetry algebra). The $N = 8$ Bianchi identities are [60, 43] (we omit the

JHEP03(2002)025

wedge product symbols among the products of forms):

$$
\begin{aligned}
R^{pq} \wedge V_q + \mathrm{i}\overline{\psi}_A \gamma^p \rho^A - \mathrm{i}\overline{\rho}_A \gamma^p \psi^A &= 0 \\
\nabla \rho_A + \frac{1}{4} R^{pq} \gamma_{pq} \psi_A - R_A{}^B \psi_B &= 0 \\
\nabla F^{\Lambda\Sigma} - 2 f^{\Lambda\Sigma}_{AB} \overline{\rho}^A \psi^B - 2 \overline{f}^{\Lambda\Sigma AB} \overline{\rho}_A \psi_B - \frac{1}{2} \overline{f}^{\Lambda\Sigma CD} P_{ABCD} \overline{\psi}^A \psi^B - \frac{1}{2} f^{\Lambda\Sigma}_{CD} \overline{P}^{ABCD} \overline{\psi}_A \psi_B &= 0 \\
\nabla \left(\nabla \chi_{ABC} \right) - 3 R_{[A}{}^L \chi_{BC]L} + \frac{1}{4} R^{pq} \gamma_{pq} \chi_{ABC} &= 0 \\
\nabla P_{ABCD} &= 0
\end{aligned}
$$

in terms of the supercovariant field-strengths:

$$
\begin{aligned}
T^p &\equiv \mathcal{D} V^p - \frac{i}{2} \; \overline{\psi}_A \gamma^p \psi^A = 0 \\
R^{pq} &\equiv d\omega^{pq} - \omega^p{}_r \omega^{rq} \\
F^{\Lambda\Sigma} &\equiv dA^{\Lambda\Sigma} + f^{\Lambda\Sigma}_{AB} \overline{\psi}^A \psi^B + \overline{f}^{\Lambda\Sigma|AB} \overline{\psi}_A \psi_B \\
\rho_A &\equiv \mathcal{D} \psi_A + \omega_A{}^B \psi_B \\
\nabla \chi_{ABC} &\equiv \mathcal{D} \chi_{ABC} + 3 \omega_{[A}{}^L \chi_{BC]L} \\
R_A{}^B &\equiv d\omega_A{}^B + \omega_A{}^C \omega_C{}^B \,.
\end{aligned}
$$

Note that all the fields are actually superfield 1-forms whose restriction at $\theta = d\theta = 0$ gives the ordinary space-time fields.

To show how the Bianchi identities of the $N = 8$ theory reduce to the Bianchi identities of the $N = N'$ theory, we just work out the example of the $N = 8 \to N = 6$ reduction. The other cases can be analyzed in analogous way.

First of all we see that, by decomposing the R-symmetry indices as in section 2 and setting $\psi_i = 0$ $(i = 7, 8)$, the supercovariant field-strengths get reduced as follows: the superspace bosonic vielbein V^p and the spin connection ω^{pq} $(p, q$ denote space-time flat indices) remain untouched by the reduction, and the same happens of course for the Lorentz curvature R^{pq} and supertorsion T^p.

As far as the gravitinos are concerned, we find:

$$
\begin{aligned}
\rho_a &\equiv \mathcal{D} \psi_a + \omega_a{}^b \psi_b \\
0 &= \rho_i = \omega_i{}^a \psi_a
\end{aligned} \tag{A.1}
$$

which implies $\omega_i{}^a = 0$, consistently with what we found in section 4. As a consequenc, the gravitinos Bianchi identities reduce to:

$$
\nabla \rho_a + \frac{1}{4} R^{pq} \gamma_{pq} \rho_a + R_a{}^b \psi_b = 0 \tag{A.2}
$$

which is the correct Bianchi identity for an $N = 6$ gravitino, while consistency of the truncation implies:

$$
\nabla \rho_i = R_i{}^a \psi_a = 0 \quad \to R_i{}^a = 0 \tag{A.3}
$$

again in agreement with the σ-model results.

JHEP03(2002)025

Let us analyze the spin one-half sector. It gives

$$
\begin{aligned}
\nabla\chi_{abc} &= \mathcal{D}\chi_{abc} + 3\omega_{[a}{}^{d}\chi_{bc]d} + \omega_{[a}{}^{i}\chi_{bc]i} \\
\nabla\chi_{abi} &= \mathcal{D}\chi_{abi} - 2\omega_{[a}{}^{d}\chi_{b]di} + 2\omega_{[a}{}^{j}\chi_{b]ij} + \omega_{i}{}^{d}\chi_{dab} + \omega_{i}{}^{j}\chi_{jab} \\
\nabla\chi_{aij} &= \mathcal{D}\chi_{aij} - \omega_{a}{}^{d}\chi_{dij} + 2\omega_{[i}^{\ d}\chi_{j]ad} + \omega_{i}{}^{dj}\chi_{jab}\,. \qquad \text{(A.4)}
\end{aligned}
$$

Since $\omega_a{}^i = 0$, we see that the last equation is satisfied only setting $\chi_{iab} = 0$, (as already known from section 2, since they belong to the gravitino multiplets truncated out). What is left is the spin one-half sector of the $N = 6$ theory. It is now straightforward to see that the Bianchi identities for χ_{abc} and χ_{aij} reduce, after imposing again the constraint $\omega_a{}^i = 0$, to the corresponding $N = 6$ Bianchi identities, while the Bianchi identity for χ_{abi} is, consistently, identically satisfied.

The analysis of the scalar sector P_{ABCD} and its Bianchi identity is identical to what has been already discussed in section 4, and does not deserve further analysis.

Finally, the Bianchi identity for the vector field strengths, with $\psi_i = \rho_i = 0$, reduces to:

$$
\nabla F^{\Lambda\Sigma} = -2f^{\Lambda\Sigma}_{ab}\overline{\rho}^a\psi^b - \frac{1}{2}\overline{f}^{\Lambda\Sigma|cd}P_{abcd}\overline{\psi}^a\psi^b - \frac{1}{2}\overline{f}^{\Lambda\Sigma|ij}P_{abij}\overline{\psi}^a\psi^b - \frac{1}{2}\overline{f}^{\Lambda\Sigma|ci}P_{abci}\overline{\psi}^a\psi^b\,. \qquad \text{(A.5)}
$$

Here, the scalar vielbein $P_{abci} = 0$ according to the discussion of section 2 and 4. Furthermore, the reduction of the couple of indices $\Lambda\Sigma$ goes according to what we have discussed in section 5. Since the duality group acts now on the electric and magnetic field-strengths in the representation **32** of $\mathrm{SO}^*(12)$, we simply substitute the couple $\Lambda\Sigma$ with an index r running from 1 to 16. Note that the corresponding quantities f^r_{ab}, f^r_{ij} are 16×16 sub-blocks of the 32×32 matrix U, which has exactly the same form of eq. (5.2), but valued in $\mathrm{Sp}(32,\mathbb{R})$, which gives the embedded coset representative.

B. Consistency of the Bianchi identities for $N = 2 \to N = 1$ gauged theory in $D = 4$

In the same spirit of the analysis of section 5, it is easy to show that the closure of Bianchi identities of the $N = 2$ theory implies the consistent closure of the reduced $N = 1$ theory.

The definition of the supercurvatures and superspace Bianchi identities for the $N = 2$ theory have been given in ref [5] (appendix A).

We have to reduce these objects to their $N = 1$ expressions, and to show that they coincide with the definitions of the supercurvatures and superspace Bianchi identities for the $N = 1$ theory. We quote in the following their standard expression.

Curvatures of $N = 1$ gauged theory.

$$
\begin{aligned}
T^a &\equiv \mathcal{D}V^a - \mathrm{i}\overline{\psi}^{\bullet}\gamma^a\psi_{\bullet} \equiv 0 \\
R^{ab} &= d\omega^{ab} - \omega^a{}_c\omega^{cb} \\
\rho_{\bullet} &= \nabla\psi_{\bullet} = \mathcal{D}\psi_{\bullet} + \frac{\mathrm{i}}{2}\widehat{Q}\psi_{\bullet}
\end{aligned}
$$

JHEP03(2002)025

$$
\begin{aligned}
R\left(\chi^{i}\right) &= \widehat{\nabla}\chi^{i} = \mathcal{D}\chi^{i} + \widehat{\Gamma}^{i}{}_{j}\chi^{j} - \frac{\mathrm{i}}{2}\widehat{Q}\chi^{i} \\
F^{\Lambda} &= dA^{\Lambda} + \frac{1}{2}C^{\Lambda}{}_{\Sigma\Gamma}A^{\Sigma}A^{\Gamma} \\
\nabla\lambda^{\Lambda} &= \mathcal{D}\lambda^{\Lambda} + \frac{\mathrm{i}}{2}\widehat{Q}\lambda^{\Lambda} + C^{\Lambda}{}_{\Sigma\Gamma}A^{\Sigma}\lambda^{\Gamma} \\
\nabla z^{i} &= dz^{i} + g_{(\Lambda)}k^{i}_{\Lambda}A^{\Lambda} ,
\end{aligned} \tag{B.1}
$$

where the gauged connections are defined as:

$$
\begin{aligned}
\widehat{\Gamma}^{i}{}_{j} &= \Gamma^{i}{}_{j} + g_{(\Lambda)}\nabla_{j}k^{i}_{\Lambda}A^{\Lambda} \\
\widehat{Q} &= Q + g_{(\Lambda)}P_{\Lambda}A^{\Lambda} .
\end{aligned} \tag{B.2}
$$

The ungauged connection Q is given by

$$
Q = Q_{i}\nabla z^{i} + Q_{\bar{\imath}}\nabla\bar{z}^{\bar{\imath}} . \tag{B.3}
$$

Bianchi identities of $N = 1$ gauged theory.

$$
\begin{aligned}
R^{ab}V_{b} - \mathrm{i}\overline{\psi}^{\bullet}\gamma^{a}\rho_{\bullet} + \mathrm{i}\overline{\psi}_{\bullet}\gamma^{a}\rho^{\bullet} &= 0 \\
\mathcal{D}R^{ab} &= 0 \\
\nabla^{2}\psi_{\bullet} + \frac{1}{4}\gamma_{ab}R^{ab}\psi_{\bullet} - \frac{\mathrm{i}}{2}\widehat{\mathcal{K}}\psi_{\bullet} &= 0 \\
\nabla^{2}\chi^{i} + \frac{1}{4}\gamma_{ab}R^{ab}\chi^{i} + \widehat{R}^{i}{}_{j}\chi^{j} + \frac{\mathrm{i}}{2}\widehat{\mathcal{K}}\chi^{i} &= 0 \\
\nabla F^{\Lambda} &= 0 \\
\nabla^{2}\lambda^{\Lambda} + \frac{1}{4}\gamma_{ab}R^{ab}\lambda^{\Lambda} - \frac{\mathrm{i}}{2}\widehat{\mathcal{K}}\lambda^{\Lambda} - C^{\Lambda}{}_{\Sigma\Gamma}A^{\Sigma}\lambda^{\Gamma} &= 0 \\
\nabla^{2}z^{i} - g_{(\Lambda)}k^{i}_{\Lambda}F^{\Lambda} &= 0 .
\end{aligned} \tag{B.4}
$$

In the ungauged case, it is straightforward to see that the conditions found in the text from the analysis of the reduction of the quaternionic sector and of supersymmetry transformation laws are indeed necessary and sufficient, after setting $\psi_2 = \rho_2 = 0$, for reducing the $N = 2$ supercurvatures and Bianchi identities to the corresponding $N = 1$ expressions.

We only observe that in the covariant differential of ζ_α and its Bianchi identity, after decomposition of the index $\alpha = (I, \dot{I})$, we get, as integrability condition:

$$
\nabla^{2}\zeta_{I} + \frac{1}{4}R^{ab}\gamma_{ab}\zeta_{I} + \frac{\mathrm{i}}{2}K\zeta_{I} + \mathbb{R}_{I}{}^{J}\zeta_{J} = 0 , \qquad (\text{since } \mathbb{R}_{I}{}^{\dot{J}} = 0). \tag{B.5}
$$

This equation can be converted in world indices on $\mathcal{M}^{KH}$ using equation (6.146). Using further the reduction of eq. (6.140) one then recovers the correct $N = 1$ result, in terms of the Riemann curvature of the $\mathcal{M}^{KH}$ manifold, with

$$
\mathcal{K}^{(N=1)} = \mathcal{K}^{(N=2)} + \Omega^{3} . \tag{B.6}
$$

Note that Ω^3 is one half of the Kähler form of the Kähler-Hodge manifold $\mathcal{M}^{KH}$.

JHEP03(2002)025

As far as the gauged theory is concerned, we observe that the ungauged conditions

$$\Gamma^{\alpha}{}_{i} = R^{\alpha}{}_{i} = \omega^1 = \omega^2 = \Omega^1 = \Omega^2 = \Delta_I{}^{j} = R_I{}^{j} = 0 \tag{B.7}$$

become the corresponding ones for the gauged quantities

$$\widehat{\Gamma}^{\alpha}{}_{i} = \widehat{R}^{\alpha}{}_{i} = \widehat{\omega}^1 = \widehat{\omega}^2 = \widehat{\Omega}^1 = \widehat{\Omega}^2 = \widehat{\Delta}_I{}^{j} = \widehat{R}_I{}^{j} = 0\,. \tag{B.8}$$

Recalling the definition of the hatted quantities, we find that the following objects must be zero:

$$\begin{aligned} g_{(\mathbf{\Lambda})} A^{\mathbf{\Lambda}} \mathcal{D}_j k^{\alpha}_{\mathbf{\Lambda}} &= g_{(\mathbf{\Lambda})} F^{\mathbf{\Lambda}} \mathcal{D}_{\mathcal{J}} k^{\alpha}_{\mathbf{\Lambda}} = 0 \\ g_{(\mathbf{\Lambda})} A^{\mathbf{\Lambda}} (P^1_{\mathbf{\Lambda}} - \mathrm{i} P^2_{\mathbf{\Lambda}}) &= g_{(\mathbf{\Lambda})} F^{\mathbf{\Lambda}} (P^1_{\mathbf{\Lambda}} - \mathrm{i} P^2_{\mathbf{\Lambda}}) = 0 \\ g_{(\mathbf{\Lambda})} A^{\mathbf{\Lambda}} \partial_u k^{v}_{\mathbf{\Lambda}} \mathcal{U}^{u|AI} \mathcal{U}_{v|Aj} &= 0\,. \end{aligned} \tag{B.9}$$

The previous conditions can be analyzed in the light of the results obtained in section 6, and it is straightforward to see that they are actually satisfied. Thus the reduced theory has the correct integrability conditions.

C. A useful formula for the $N = 2$ gaugino transformation law

In this appendix we show how to retrieve equation (6.80) from (6.3). To avoid a too heavy notation, we write in this appendix the world indices and gauge indices without hat and tilde, since we are not going to perform any reduction. We are interested in trading the world index i of the gauginos λ^{iA} with a gauge index Λ, through the definition:

$$\lambda^{\Lambda A} \equiv -2 f^{\Lambda}_{i} \lambda^{iA}\,. \tag{C.1}$$

However, the gauge index of the $N = 2$ theory runs over $n_V + 1$ values (because of the presence of the graviphoton) while the index i takes only n_V values. The extra gaugino, say λ^0, is actually spurious, since, as discussed in section 6.2, $\lambda^{\Lambda A}$ satisfies:

$$T_{\Lambda} \lambda^{\Lambda A} = 0\,, \tag{C.2}$$

where T_Λ is the projector on the graviphoton field-strength, according to equation (6.15) [48]. In order to show that the n_V gauginos λ^{Λ} do appropriately transform into the n_V matter-vector field strengths, let us now calculate the susy transformation law of the new fermions $\lambda^{\Lambda A}$, which, up to 3-fermions terms, is:

$$\delta\lambda^{\Lambda A} = -2 f^{\Lambda}_{i} \delta\lambda^{iA} = -2 f^{\Lambda}_{i} \left[-g^{i\bar{\jmath}} f^{\Sigma}_{\bar{\jmath}} \mathrm{Im}\mathcal{N}_{\Gamma\Sigma} F^{-\Gamma}_{\mu\nu} \gamma^{\mu\nu} \varepsilon^{AB} + W^{iAB} \right] \epsilon_B\,. \tag{C.3}$$

Now we use the following relations of special geometry [48]:

$$g_{i\bar{\jmath}} = -2 f^{\Lambda}_{i} \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f^{\Sigma}_{\bar{\jmath}} \tag{C.4}$$

$$\delta^{i}_{\ell} = g^{i\bar{\jmath}} g_{\ell\bar{\jmath}} = -2 g^{i\bar{\jmath}} f^{\Lambda}_{\ell} \mathrm{Im}\mathcal{N}_{\Lambda\Sigma} f^{\Sigma}_{\bar{\jmath}} \tag{C.5}$$

$$U^{\Lambda\Sigma} \equiv f^{\Lambda}_{i} g^{i\bar{\jmath}} f^{\Sigma}_{\bar{\jmath}} = -\frac{1}{2} \left[(\mathrm{Im}\mathcal{N})^{-1} \right]^{\Lambda\Sigma} - \overline{L}^{\Lambda} L^{\Sigma} \tag{C.6}$$

$$\mathrm{Im}\mathcal{N}_{\Lambda\Sigma} L^{\Lambda} \overline{L}^{\Sigma}_{i} = -\frac{1}{2}\,. \tag{C.7}$$

JHEP03(2002)025

Eq. (C.3) can then be rewritten as:

$$\delta\lambda^{\Lambda A} = \left[2U^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Sigma\Gamma}F^{-\Gamma}_{\mu\nu}\varepsilon^{AB} - 2f^{\Lambda}_{\ell}k^{\ell}_{\Delta}\overline{L}^{\Delta}\varepsilon^{AB} - 2\mathrm{i}U^{\Lambda\Sigma}P^{x}_{\Sigma}(\sigma^{x})^{AB}\right]\epsilon_{B}\,. \tag{C.8}$$

Now, using the definition of the special geometry Killing vectors

$$k^{i}_{\Lambda} = \mathrm{i}g^{i\overline{\jmath}}\partial_{\overline{\jmath}}P^{0}_{\Lambda} \tag{C.9}$$

we have:

$$2f^{\Lambda}_{\ell}k^{\ell}_{\Delta}\overline{L}^{\Delta} = 2\mathrm{i}f^{\Lambda}_{\ell}g^{\ell\overline{\ell}}\partial_{\overline{\ell}}P^{0}_{\Delta}\overline{L}^{\Delta} = -2\mathrm{i}f^{\Lambda}_{\ell}g^{\ell\overline{\ell}}f^{\Delta}_{\overline{\ell}}P^{0}_{\Delta} = -2\mathrm{i}U^{\Lambda\Sigma}P^{0}_{\Sigma}\,, \tag{C.10}$$

where we have used the special geometry formulae [5]:

$$P^{0}_{\Lambda}L^{\Lambda} = P^{0}_{\Lambda}\overline{L}^{\Lambda} = 0\,, \quad f^{\Lambda}_{i} \equiv \nabla_{i}L^{\Lambda}\,. \tag{C.11}$$

Therefore eq. (C.8) becomes:

$$\delta\lambda^{\Lambda A} = -2U^{\Lambda\Sigma}\left[\mathrm{Im}\mathcal{N}_{\Sigma\Gamma}F^{-\Gamma}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB} + \mathrm{i}\left(-P^{0}_{\Sigma}\varepsilon^{AB} + P^{x}_{\Sigma}(\sigma^{x})^{AB}\right)\right]\epsilon_{B}\,. \tag{C.12}$$

Let us now set

$$P^{\Lambda}{}_{\Gamma} \equiv -2U^{\Lambda\Sigma}\mathrm{Im}\mathcal{N}_{\Sigma\Gamma} = \delta^{\Lambda}_{\Gamma} + 2\mathrm{Im}\mathcal{N}_{\Gamma\Sigma}\overline{L}^{\Lambda}L^{\Sigma} = \delta^{\Lambda}_{\Gamma} - \mathrm{i}T_{\Gamma}\overline{L}^{\Lambda} \tag{C.13}$$

$$\overline{P}_{\Sigma}{}^{\Lambda} = \delta^{\Lambda}_{\Sigma} + \mathrm{i}\overline{T}_{\Sigma}L^{\Lambda} = (P^{t})^{\Lambda}{}_{\Sigma}\,, \tag{C.14}$$

where T_Λ is defined by equation (6.78) and satisfies [48] :

$$T_{\Lambda}\overline{L}^{\Lambda} = -\mathrm{i}\,; \qquad T_{\Lambda}f^{\Lambda}_{i} = 0\,. \tag{C.15}$$

Then we have:

$$P^{\Lambda}{}_{\Sigma}P^{\Sigma}{}_{\Gamma} = P^{\Lambda}{}_{\Gamma}\,; \quad T_{\Lambda}P^{\Lambda}_{\Gamma} = 0\,. \tag{C.16}$$

Therefore $P^{\Lambda}{}_{\Gamma}$ is the projector orthogonal to the graviphoton, that is it projects the n_v+1 vector field-strengths onto the n_v field-strengths of the vector multiplets.

We can then rewrite equation (C.12) as:

$$\delta\lambda^{\Lambda A} = P^{\Lambda}{}_{\Sigma}F^{-\Sigma}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB} + \mathrm{i}U^{\Lambda\Gamma}\left(-P^{0}_{\Gamma}\varepsilon^{AB} + P^{x}_{\Gamma}(\sigma^{x})^{AB}\right)\epsilon_{B} \tag{C.17}$$

which is the equation given in the text. Note that

$$\lambda^{\Lambda A} = P^{\Lambda}{}_{\Sigma}\lambda^{\Sigma A}\,; \qquad f^{\Lambda}_{i} = P^{\Lambda}{}_{\Sigma}f^{\Sigma}_{i}\,. \tag{C.18}$$

It is useful to write down the explicit decomposition of the field strength F^Λ into the graviphoton and matter vectors part, that is:

$$F^{-\Lambda}_{\mu\nu} = \mathrm{i}\overline{L}^{\Lambda}T_{\Sigma}F^{-\Sigma}_{\mu\nu} + P^{\Lambda}{}_{\Sigma}F^{-\Sigma}_{\mu\nu}\,. \tag{C.19}$$

Eq. (C.17) becomes:

$$\delta\lambda^{\Lambda}_{\bullet} = P^{\Lambda}{}_{\Sigma}\left[F^{-\Gamma}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{AB} + \mathrm{i}\,(\mathrm{Im}\mathcal{N})^{-1\Sigma\Gamma}\left(-P^{0}_{\Gamma}\varepsilon^{AB} + P^{x}_{\Gamma}(\sigma^{x})^{AB}\right)\right]\epsilon_{B}\,, \tag{C.20}$$

where we see, as expected, that the gauginos $\lambda^{\Lambda A}$ do transform only into the matter-vector field strengths $P^{\Lambda}{}_{\Sigma}F^{-\Sigma}_{\mu\nu}$. Hence, equation (C.20) intrinsically defines only n_V independent gauginos transforming into the $N=2$ field strengths $(P^{\Lambda}{}_{\Sigma}F^{-\Sigma}_{\mu\nu})$.

JHEP03(2002)025

D. Reduction of special geometry in special coordinates

If we choose special coordinates for special geometry [3, 9, 48, 59], then the indices $\mathbf{\Lambda} = (\Lambda, Y)$ and $\mathcal{I} = (i, \alpha)$ are identified by the fact that

$$t^{\mathcal{I}} = \frac{X^{\mathbf{\Lambda}}}{X^0}\,, \quad (\Lambda = \alpha\,,\, Y = i) \tag{D.1}$$

and a prepotential $F(X)$ exists such that $f(t) = \frac{1}{(X^0)^2} F(X)$, with:

$$\begin{aligned} X^0 F_0 &= 2f - t^{\mathcal{I}} f_{\mathcal{I}}\,, \quad \left(f_{\mathcal{I}} = \frac{\partial f}{\partial t^{\mathcal{I}}}\right) \\ X^0 F_{\mathcal{I}} &= \partial_{\mathcal{I}} f\,. \end{aligned} \tag{D.2}$$

Furthermore,

$$e^{-\mathcal{K}} = \mathrm{i}\left[2f - 2\overline{f} + (\overline{t}^{\mathcal{I}} - t^{\mathcal{I}})(f_{\mathcal{I}} + \overline{f}_{\mathcal{I}})\right]. \tag{D.3}$$

The constraints that define the submanifold $\mathcal{M}_R$ become:

$$\begin{aligned} W_{ij\alpha} &= \partial_i\partial_j\partial_\alpha f = 0\,, \quad W_{\alpha_1\alpha_2\alpha_3} = \partial_{\alpha_1}\partial_{\alpha_2}\partial_{\alpha_3} f = 0 \\ X^\Lambda &= \frac{\partial F}{\partial X^\Lambda} = \partial_i X^\Lambda = \partial_\alpha X^X = \partial_i f_\Lambda = \partial_\alpha f_X = 0\,, \end{aligned} \tag{D.4}$$

where we used the fact that $\mathcal{K}_\alpha|_{\mathcal{M}_R} = 0$.

In this basis $\overline{\mathcal{N}}_{\mathbf{\Lambda}\Sigma} = \partial_{\mathbf{\Lambda}}\partial_\Sigma f$ and the Kähler potential on $\mathcal{M}_R$ is:

$$e^{-\mathcal{K}_R} = \mathrm{i}\left[2f - 2\overline{f} + (\overline{t}^i - t^i)(f_i + \overline{f}_i)\right]. \tag{D.5}$$

Note that $F_\Lambda|_{\mathcal{M}_R} = 0$ implies $\partial_\alpha f|_{\mathcal{M}_R} = 0$ which in turn implies $\partial_\alpha\partial_i f|_{\mathcal{M}_R} = 0$, $W_{\alpha ij} = \partial_\alpha\partial_i\partial_j f|_{\mathcal{M}_R} = 0$. Therefore the most general form for f is ($t^{\mathcal{I}} \Rightarrow (t^i, z^\alpha)$):

$$f(t^i, z^\alpha) = f(t) + \sum_{n\geq 2} z^{\alpha_1}\cdots z^{\alpha_n} f_{\alpha_1\cdots\alpha_n}(t)\,, \quad f_{\alpha_1\alpha_2\alpha_3}(t) = 0\,. \tag{D.6}$$

For the manifold $\mathrm{SU}(1,1)/\,\mathrm{U}(1) \times \mathrm{SO}(2,n)/[\mathrm{SO}(2)\times\mathrm{SO}(n)]$ used in section 6.6, with coordinates $(t^0, t^1, \ldots, t^{n'}, z^1, \ldots, z^{n-n'})$, the reduced manifold $(z^1, \ldots, z^{n-n'}) = 0$ is parametrized with coordinates $(t^0, t^1, \ldots, t^{n'})$ and the holomorphic prepotential is [55, 48]:

$$f(t^i, z^\alpha) = \mathrm{i}t^0\left(\sum_{i=1}^{n'} \eta_{ij} t^i t^j - \sum_{i=1}^{n-n'} \delta_{\alpha\beta} z^\alpha z^\beta\right), \quad [\eta_{ij} = (1, -1, \ldots, -1)] \tag{D.7}$$

in accordance to equation (D.6).

E. Reduction of the quaternionic Ward identity

We derive here the conditions on the quaternionic prepotentials and Killing vectors, discussed in section 6.5, from the reduction of the quaternionic Ward identity (6.176), which

JHEP03(2002)025

is essential for the validity of the $N = 2$ supersymmetric Ward identity involving the scalar potential, that is the relation [5]:

$$\frac{1}{\lambda}\Omega^x_{uv} k^u_{\boldsymbol{\Lambda}} k^v_{\boldsymbol{\Sigma}} + \frac{1}{2}\epsilon^{xyz} P^y_{\boldsymbol{\Lambda}} P^z_{\boldsymbol{\Sigma}} - \frac{1}{2} f_{\boldsymbol{\Lambda}\boldsymbol{\Sigma}}{}^{\boldsymbol{\Gamma}} P^x_{\boldsymbol{\Gamma}} = 0\,. \tag{E.1}$$

After projection, and using the just found results $P^{\mathrm{i}}_\Lambda = 0$; $P^3_X = 0$, it decomposes in a set of equations

- $\boldsymbol{\Lambda} = \Lambda$; $\boldsymbol{\Sigma} = \Sigma$

$$\frac{1}{\lambda}\Omega^{\mathrm{i}}_{uv} k^u_\Lambda k^v_\Sigma + \frac{1}{2}\epsilon^{\mathrm{ij}} P^{\mathrm{j}}_\Lambda P^3_\Sigma - \frac{1}{2} f_{\Lambda\Sigma}{}^{\boldsymbol{\Gamma}} P^{\mathrm{i}}_{\boldsymbol{\Gamma}} = 0 \tag{E.2}$$

$$\frac{1}{\lambda}\Omega^3_{uv} k^u_\Lambda k^v_\Sigma + \frac{1}{2}\epsilon^{\mathrm{ij}} P^{\mathrm{i}}_\Lambda P^{\mathrm{j}}_\Sigma - \frac{1}{2} f_{\Lambda\Sigma}{}^{\boldsymbol{\Gamma}} P^3_{\boldsymbol{\Gamma}} = 0\,. \tag{E.3}$$

Since $P^{\mathrm{j}}_\Lambda = 0$, and since $f_{\Lambda\Sigma}{}^Z = 0$ (because $G^{(1)} \subset G^{(2)}$), then eq. (E.2) gives

$$k^t_\Lambda = 0\,, \tag{E.4}$$

as indeed was expected from geometrical considerations.

Then, equation (E.3) becomes

$$\frac{1}{\lambda}\Omega^3_{s\bar{s}} k^s_\Lambda k^{\bar{s}}_\Sigma - \frac{1}{2} f_{\Lambda\Sigma}{}^{\Gamma} P^3_\Gamma = 0\,. \tag{E.5}$$

Setting $\lambda = -1$ and recalling that $-\Omega_{s\bar{s}}$ is half of the Kähler form of the reduced submanifold, we recognize that eq. (E.5) expresses the poissonian realization of the Lie algebra of the prepotentials P^3_Λ on the Kähler-Hodge submanifold $\mathcal{M}^{KH}$, namely:

$$\{P^3_\Lambda, P^3_\Sigma\}_P = f_{\Lambda\Sigma}{}^{\Gamma} P^3_\Gamma\,. \tag{E.6}$$

- $\boldsymbol{\Lambda} = \Lambda$; $\boldsymbol{\Sigma} = Y$

$$\frac{1}{\lambda}\Omega^{\mathrm{i}}_{st} k^s_\Lambda k^t_Y - \frac{1}{2}\epsilon^{\mathrm{ij}} P^3_\Lambda P^{\mathrm{j}}_Y - \frac{1}{2} f_{\Lambda Y}{}^{Z} P^{\mathrm{i}}_Z = 0 \tag{E.7}$$

$$\frac{1}{\lambda}\Omega^3_{uv} k^u_\Lambda k^v_Y + \frac{1}{2}\epsilon^{\mathrm{ij}} P^{\mathrm{i}}_\Lambda P^{\mathrm{j}}_Y - \frac{1}{2} f_{\Lambda Y}{}^{\boldsymbol{\Gamma}} P^3_{\boldsymbol{\Gamma}} = 0\,. \tag{E.8}$$

Eq. (E.7) gives a relation which has to be valid everywhere on the submanifold.

Since $\Omega^3_{st} = 0$, $P^3_X = P^{\mathrm{i}}_\Lambda = 0$, and considering (E.4), then eq. (E.8) is identically satisfied.

- $\boldsymbol{\Lambda} = X$; $\boldsymbol{\Sigma} = Y$

$$\frac{1}{\lambda}\Omega^{\mathrm{i}}_{st} k^s_X k^t_Y - \frac{1}{2}\epsilon^{\mathrm{ij}} P^3_X P^{\mathrm{j}}_Y - \frac{1}{2} f_{XY}{}^{Z} P^{\mathrm{i}}_Z = 0 \tag{E.9}$$

$$\frac{1}{\lambda}\Omega^3_{tt'} k^t_X k^{t'}_Y + \frac{1}{2}\epsilon^{\mathrm{ij}} P^{\mathrm{i}}_X P^{\mathrm{j}}_Y - \frac{1}{2} f_{XY}{}^{\Gamma} P^3_\Gamma = 0\,. \tag{E.10}$$

Eq. (E.9) is identically satisfied for $f_{XY}{}^Z$, while eq. (E.10) is a relation to be satisfied all over the submanifold.

JHEP03(2002)025

F. Computation of the $N = 2 \to N = 1$ scalar potential

We want to solve here a puzzle raised in the text about the scalar potential. In the $N = 2$ theory, the scalar potential has the form [5]:

$$\begin{aligned}\mathcal{V}^{N=2} &= -12 S^{11}\overline{S}_{11} + g_{\mathcal{I}\overline{\mathcal{J}}}\left(W^{\mathcal{I}11}\overline{W}^{\overline{\mathcal{J}}}_{11} + W^{\mathcal{I}21}\overline{W}^{\overline{\mathcal{J}}}_{21}\right) + 2N^1_I N^I_1 \\ &= \left(g_{\mathcal{I}\overline{\mathcal{J}}} k^{\mathcal{I}}_{\mathbf{\Lambda}} k^{\overline{\mathcal{J}}}_{\mathbf{\Sigma}} + 4h_{uv} k^u_{\mathbf{\Lambda}} k^v_{\mathbf{\Sigma}}\right)\overline{L}^{\mathbf{\Lambda}} L^{\mathbf{\Sigma}} + U^{\mathbf{\Lambda\Sigma}} P^x_{\mathbf{\Lambda}} P^x_{\mathbf{\Sigma}} - 3P^x_{\mathbf{\Lambda}} P^x_{\mathbf{\Sigma}}\overline{L}^{\mathbf{\Lambda}} L^{\mathbf{\Sigma}}\,. \end{aligned} \tag{F.1}$$

which manifestly does not contain contributions antisymmetric in the quaternionic prepotentials or Killing vectors, nor it has interference terms $P^0_{\mathbf{\Lambda}} P^x_{\mathbf{\Sigma}}$ between quaternionic and special-Kähler isometries.

On the other hand, the $N = 1$ scalar potential:

$$\mathcal{V}^{N=1} = 4\left(\nabla_\ell L \nabla_{\overline{\ell}} \overline{L} g^{\ell\overline{\ell}} - 3|L|^2 + \frac{1}{16}\mathrm{Im} f_{\Lambda\Sigma} D^\Lambda D^\Sigma\right), \tag{F.2}$$

does instead contain both kinds of interference contributions, given the form of the superpotential $L = \frac{\mathrm{i}}{2} L^X (P^1_X - \mathrm{i}P^2_X)$ and of the D-term $D^\Lambda = -2(\mathrm{Im} f)^{-1\Lambda\Sigma}(P^0_\Sigma + P^3_\Sigma)$ which appear quadratically in (F.2).

The interference contributions in (F.2) have therefore to cancel each other. As we are going to show, this does indeed happen, in a way which involves non trivially the properties obeyed by the special-Kähler and quaternionic Killing vectors. Let us analyze and reduce separately the various contributions to the $N = 2$ potential, using all the constraint relations found in section 6.

$$\begin{aligned} -12S^{11}\overline{S}_{11} &\Rightarrow -12|L|^2 \\ &= -3P^{\mathrm{i}}_X P^{\mathrm{i}}_Y \overline{L}^X L^Y - 6\mathrm{i} P^1_{[X} P^2_{Y]} L^X \overline{L}^Y \end{aligned} \tag{F.3}$$

$$\begin{aligned} g_{\mathcal{I}\overline{\mathcal{J}}} W^{\mathcal{I}11}\overline{W}^{\overline{\mathcal{J}}}_{11} U^{\mathbf{\Lambda\Sigma}} &\Rightarrow 4 g^{i\overline{\jmath}} \nabla_i L \nabla_{\overline{\jmath}} \overline{L} \\ &= g^{i\overline{\jmath}} \nabla_i L^X \nabla_{\overline{\jmath}} \overline{L}^Y P^{\mathrm{i}}_X P^{\mathrm{i}}_Y - 2\mathrm{i} P^1_{[X} P^2_{Y]} U^{XY} \\ &= g^{i\overline{\jmath}} \nabla_i L^X \nabla_{\overline{\jmath}} \overline{L}^Y P^{\mathrm{i}}_X P^{\mathrm{i}}_Y + 2\mathrm{i} P^1_{[X} P^2_{Y]} L^X \overline{L}^Y \end{aligned} \tag{F.4}$$

$$g_{\mathcal{I}\overline{\mathcal{J}}} W^{\mathcal{I}21}\overline{W}^{\overline{\mathcal{J}}}_{21} \Rightarrow (\mathrm{Im} f)^{-1\Lambda\Sigma}\left(P^0_\Lambda P^0_\Sigma + P^3_\Lambda P^3_\Sigma\right) + 2U^{\Lambda\Sigma} P^0_\Lambda P^3_\Sigma\,, \tag{F.5}$$

where we have used equation (C.10) of appendix C, the identity of special geometry (6.100) and the definition of the prepotential P^0_Γ, equation (6.93).

$$\begin{aligned} 2N^1_I N^I_1 &\Rightarrow 4 g^{s\overline{s}} \nabla_s L \nabla_{\overline{s}} \overline{L} \\ &= g^{s\overline{s}} \nabla_s P^{\mathrm{i}}_X \nabla_{\overline{s}} P^{\mathrm{i}}_Y + 2\mathrm{i} g^{s\overline{s}} \nabla_s P^{\mathrm{i}}_{[X} \nabla_{\overline{s}} P^{\mathrm{i}}_{Y]}\,. \end{aligned} \tag{F.6}$$

The last term in equation (F.6) is transformed using the definition of quaternionic Killing vectors:

$$2k^v_\Lambda \Omega^x_{uv} = \nabla_u P^x_\Lambda \tag{F.7}$$

the realization of the SU(2) algebra on the curvatures Ω^x:

$$h^{st}\Omega^x_{us}\Omega^y_{tw} = -\lambda^2 \delta^{xy} h_{uw} + \lambda \epsilon^{xyz}\Omega^z_{uw} \tag{F.8}$$

JHEP03(2002)025

and the normalization chosen for the metric on $\mathcal{M}^{KH}$:

$$h_{s\bar{s}} = \frac{1}{2} g_{s\bar{s}} \,. \tag{F.9}$$

After some calculations we get:

$$\begin{aligned}
2\mathrm{i}g^{s\bar{s}}\nabla_s P^{\mathrm{i}}_{[X}\nabla_{\bar{s}}P^{\mathrm{i}}_{Y]} &= 4\mathrm{i}\Omega^3_{t\bar{t}}k^t_{[X}h^{\bar{t}}_{Y]}L^X\overline{L}^Y \\
&= \left(4\mathrm{i}P^1_{[X}P^2_{Y]} - \frac{\mathrm{i}}{2}f^{\Gamma}{}_{X\dot{\Sigma}}P^3_{\Gamma}\right)L^{\dot{\Lambda}}\overline{L}^Y \\
&= 4\mathrm{i}P^1_{[X}P^2_{Y]}L^X\overline{L}^Y - 2P^3_{\Lambda}P^0_{\Sigma}(\mathrm{Im}f)^{-1\Lambda\Sigma}\,,
\end{aligned} \tag{F.10}$$

where we have applied the quaternionic Ward identity (6.176) discussed in appendix D to the present case and the definition of the prepotential P^0_Λ, eq. (6.93). Collecting together all the terms in eqs. (F.3), (F.4), (F.5) and (F.10), we find that the antisymmetric parts in Λ, Σ and the two terms in $P^0_\Lambda P^3_\Sigma$ cancel against each other identically.

G. The $N = 2$ and $N = 1$ lagrangians in $D = 4$

For reference of the reader we give here the lagrangian of the $N = 2$ theory and of the $N = 1$ theory as given in reference [45].[13]

The $N = 1$ lagrangian is, up to four-fermions terms:

$$\begin{aligned}
(\mathrm{detV})^{-1}\mathcal{L}^{\mathrm{N=1}} = &-\frac{1}{2}\mathcal{R} + \mathrm{i}\left(f_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \overline{f}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) + g_{i\bar{\jmath}}\nabla_\mu z^i\nabla^\mu z^{\bar{\jmath}} + \\
&+\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\overline{\psi}^{\bullet}_{\mu}\gamma_\sigma D_\nu\psi_{\bullet\lambda} - \overline{\psi}_{\bullet\mu}\gamma_\sigma D_\nu\psi^{\bullet}_{\lambda}\right) + \\
&+\frac{1}{8}\left(\overline{f}_{\Lambda\Sigma}\overline{\lambda}^{\bullet\Lambda}\gamma^\mu\nabla_\mu\lambda^{\Sigma}_{\bullet} - f_{\Lambda\Sigma}\overline{\lambda}^{\Lambda}_{\bullet}\gamma^\mu\nabla_\mu\lambda^{\bullet\Sigma}\right) - \\
&-\mathrm{i}\frac{1}{2}g_{i\bar{\jmath}}\left(\overline{\chi}^i\gamma^\mu\nabla_\mu\chi^{\bar{\jmath}} + \overline{\chi}^{\bar{\jmath}}\gamma^\mu\nabla_\mu\chi^i\right) - \\
&-g_{i\bar{\jmath}}\left(\overline{\psi}_{\bullet\nu}\gamma^\mu\gamma^\nu\chi^i\nabla^\mu\overline{z}^{\bar{\jmath}} + \overline{\psi}^{\bullet}_{\nu}\gamma^\mu\gamma^\nu\chi^{\bar{\jmath}}\nabla_\mu z^i\right) - \\
&-\mathrm{i}\,\mathrm{Im}f_{\Lambda\Sigma}\left(\mathcal{F}^{+\Lambda}_{\mu\nu}\overline{\lambda}^{\Sigma}_{\bullet}\gamma^\mu\psi^{\bullet\nu} + \mathcal{F}^{-\Lambda}_{\mu\nu}\overline{\lambda}^{\bullet\Sigma}\gamma^\mu\psi^{\nu}_{\bullet}\right) - \\
&-\frac{\mathrm{i}}{8}\left(\partial_i f_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\overline{\chi}^i\gamma^{\mu\nu}\lambda^{\Sigma}_{\bullet} - \partial_{\bar{\imath}}\overline{f}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\overline{\chi}^{\bar{\imath}}\gamma^{\mu\nu}\lambda^{\bullet\Sigma}\right) + \\
&+2L\overline{\psi}^{\bullet}_{\mu}\gamma^{\mu\nu}\psi^{\bullet}_{\nu} + 2\overline{L}\overline{\psi}_{\mu\bullet}\gamma^{\mu\nu}\psi_{\nu\bullet} + \\
&+\mathrm{i}g_{i\bar{\jmath}}\left(\overline{N}^{\bar{\jmath}}\overline{\chi}^i\gamma^\mu\psi^{\bullet}_{\mu} + N^i\overline{\chi}^{\bar{\jmath}}\gamma^\mu\psi_{\bullet\mu}\right) + \frac{1}{2}P_\Lambda\left(\overline{\lambda}^{\bullet\Lambda}\gamma^\mu\psi_{\bullet\mu} - \overline{\lambda}^{\Lambda}_{\bullet}\gamma^\mu\psi^{\bullet}_{\mu}\right) + \\
&+\mathcal{M}_{ij}\overline{\chi}^i\chi^j + \overline{\mathcal{M}}_{\bar{\imath}\bar{\jmath}}\overline{\chi}^{\bar{\imath}}\chi^{\bar{\jmath}} + \mathcal{M}_{\Lambda\Sigma}\overline{\lambda}^{\Lambda}_{\bullet}\lambda^{\Sigma}_{\bullet} + \overline{\mathcal{M}}_{\Lambda\Sigma}\overline{\lambda}^{\Lambda\bullet}\lambda^{\Sigma\bullet} + \\
&+\mathcal{M}_{\Lambda i}\overline{\lambda}^{\Lambda}_{\bullet}\chi^i + \overline{\mathcal{M}}_{\Lambda\bar{\imath}}\overline{\lambda}^{\Lambda\bullet}\chi^{\bar{\imath}} - \mathcal{V}(z,\overline{z},q)\,,
\end{aligned}$$

where the kinetic matrix $f_{\Lambda\Sigma}$ is a holomorphic function of z^i, and the mass matrices $\mathcal{M}_{ij}, \mathcal{M}_{\Lambda\Sigma}, \mathcal{M}_{\Lambda i}$ are given by:

$$\mathcal{M}_{ij} = \nabla_i\nabla_j L \tag{G.1}$$

[13] Some misprints of ref. [45] have been corrected

JHEP03(2002)025

$$
\begin{aligned}
\mathcal{M}_{\Lambda\Sigma} &= \frac{\mathrm{i}}{8} N^i \partial_i \overline{\mathcal{N}}_{\Lambda\Sigma} && \text{(G.2)}\\
M_{\Lambda i} &= -\mathrm{i}\frac{1}{4}\mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\partial_i D^\Sigma - \frac{1}{2} k_\Lambda^{\bar\jmath} g_{i\bar\jmath}\,, && \text{(G.3)}
\end{aligned}
$$

where we have set $\mathcal{F}^{\pm\Lambda}_{\mu\nu} = \frac{1}{2}(\mathcal{F}^\Lambda_{\mu\nu} \pm \frac{\mathrm{i}}{2}\epsilon^{\mu\nu\rho\sigma}\mathcal{F}^\Lambda_{\rho\sigma})$, $\mathcal{F}^\Lambda_{\mu\nu}$ being the field-strengths of the vectors A^Λ_μ.

Note that, since the scalar manifold is a Kähler-Hodge manifold, all the fields and the bosonic sections have a definite U(1) weight p under U(1). We have

$$
\begin{aligned}
p(V^a_\mu) &= p(A^\Lambda) = p(z^i) = p(g_{i\bar\jmath}) = p(\mathcal{N}_{\Lambda\Sigma}) = p(D^\Lambda) = p(P_\Lambda) = p(\mathcal{V}) = 0\\
p(\psi_\bullet) &= p(\chi^{\bar\imath}) = p(\lambda^\Lambda_\bullet) = p(\varepsilon_\bullet) = \frac{1}{2}\\
p(\psi^\bullet) &= p(\chi^i) = p(\lambda^{\Lambda\bullet}) = p(\varepsilon^\bullet) = -\frac{1}{2}\\
p(L) &= p(\mathcal{M}_{ij}) = p(\overline{\mathcal{M}}_{\Lambda\Sigma}) = 1\\
p(\overline{L}) &= p(\overline{\mathcal{M}}_{\bar\imath\bar\jmath}) = p(\mathcal{M}_{\Lambda\Sigma}) = -1\,. && \text{(G.4)}
\end{aligned}
$$

Accordingly, when a covariant derivative acts on a field Φ of weight p it is also U(1) covariant (besides possibly Lorentz, gauge and scalar manifold coordinate symmetries) according to the following definitions:

$$
\nabla_i\Phi = \left(\partial_i + \frac{1}{2}p\partial_i\mathcal{K}\right)\Phi;\, \nabla_{i^*}\Phi = \left(\partial_{i^*} - \frac{1}{2}p\partial_{i^*}\mathcal{K}\right)\Phi\,, \tag{G.5}
$$

where $\mathcal{K}(z,\overline{z})$ is the Kähler potential.

On the other hand, the $N=2$ lagrangian, up to four-fermions terms, is:

$$
\begin{aligned}
(\det V)^{-1}\,\mathcal{L}^{N=2} = &-\frac{1}{2}R + g_{i\bar\jmath}\nabla^\mu z^i \nabla_\mu \overline{z}^{\bar\jmath} + h_{uv}\nabla_\mu q^u \nabla^\mu q^v +\\
&+\frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}\left(\overline{\psi}^A_\mu\gamma_\sigma\rho_{A\nu\lambda} - \overline{\psi}_{A\mu}\gamma_\sigma\rho^A_{\nu\lambda}\right) -\\
&-\frac{\mathrm{i}}{2}g_{i\bar\jmath}\left(\overline{\lambda}^{iA}\gamma^\mu\nabla_\mu\lambda^{\bar\jmath}_A + \overline{\lambda}^{\bar\jmath}_A\gamma^\mu\nabla_\mu\lambda^{iA}\right) - \mathrm{i}\left(\overline{\zeta}^\alpha\gamma^\mu\nabla_\mu\zeta_\alpha + \overline{\zeta}_\alpha\gamma^\mu\nabla_\mu\zeta^\alpha\right) +\\
&+2\mathrm{i}\left(\overline{\mathcal{N}}_{\Lambda\Sigma}\mathcal{F}^{-\Lambda}_{\mu\nu}\mathcal{F}^{-\Sigma\mu\nu} - \mathcal{N}_{\Lambda\Sigma}\mathcal{F}^{+\Lambda}_{\mu\nu}\mathcal{F}^{+\Sigma\mu\nu}\right) +\\
&+\Big\{ - g_{i\bar\jmath}\nabla_\mu\overline{z}^{\bar\jmath}\overline{\psi}^\mu_A\lambda^{iA} - 2\mathcal{U}^{A\alpha}_u\nabla_\mu q^u\overline{\psi}^\mu_A\zeta_\alpha + g_{i\bar\jmath}\nabla_\mu\overline{z}^{\bar\jmath}\overline{\lambda}^{iA}\gamma^{\mu\nu}\psi_{A\nu} +\\
&\quad + 2\mathcal{U}^{\alpha A}_u\nabla_\mu q^u\overline{\zeta}_\alpha\gamma^{\mu\nu}\psi_{A\nu} + \mathrm{h.c.}\Big\} +\\
&+\Bigg\{\mathcal{F}^{-\Lambda}_{\mu\nu}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}\Bigg[4L^\Sigma\overline{\psi}^{A\mu}\psi^{B\nu}\epsilon_{AB} - 4\mathrm{i}\overline{f}^\Sigma_{\bar\imath}\overline{\lambda}^{\bar\imath}_A\gamma^\nu\psi^\mu_B\epsilon^{AB} +\\
&\qquad + \frac{1}{2}\nabla_i f^\Sigma_j\overline{\lambda}^{iA}\gamma^{\mu\nu}\lambda^{jB}\epsilon_{AB} - L^\Sigma\overline{\zeta}_\alpha\gamma^{\mu\nu}\zeta_\beta C^{\alpha\beta}\Bigg] + \mathrm{h.c.}\Bigg\} +\\
&+\Big[\mathrm{i}g_{i\bar\jmath}W^{iAB}\overline{\lambda}^{\bar\jmath}_A\gamma_\mu\psi^\mu_B + 2\mathrm{i}N^A_\alpha\overline{\zeta}^\alpha\gamma_\mu\psi^\mu_A +\\
&\quad + \mathcal{M}^{\alpha\beta}\overline{\zeta}_\alpha\zeta_\beta + \mathcal{M}^\alpha{}_{iB}\overline{\zeta}_\alpha\lambda^{iB} + \mathcal{M}_{ijAB}\overline{\lambda}^{iA}\lambda^{jB} + \mathrm{h.c.}\Big] - \mathcal{V}(z,\overline{z},q)\,. && \text{(G.6)}
\end{aligned}
$$

JHEP03(2002)025

Furthermore $L^\Lambda(z,\overline{z})$ are the covariantly holomorphic sections of the special geometry, $f_i^\Lambda \equiv \nabla_i L^\Lambda$ and the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ is constructed in terms of L^Λ and its magnetic dual according to reference [5]. The normalization of the kinetic term for the quaternions depends on the scale λ of the quaternionic manifold for which we have chosen the value $\lambda=-1$. Finally, the mass matrices of the spin $\frac{1}{2}$ fermions $\mathcal{M}^{\alpha\beta}$, $\mathcal{M}_{AB\,ij}$, $\mathcal{M}^{\alpha}_{iA}$ (and their hermitian conjugates) and the scalar potential $\mathcal{V}$ are given by:

$$\mathcal{M}^{\alpha\beta} = -\mathcal{U}^{\alpha A}_u \mathcal{U}^{\beta B}_v \,\varepsilon_{AB}\, \nabla^{[u} k^{v]}_\Lambda L^\Lambda \tag{G.7}$$

$$\mathcal{M}^{\alpha}{}_{iB} = -4\,\mathcal{U}^{\alpha}_{Bu}\, k^u_\Lambda\, f_i^\Lambda \tag{G.8}$$

$$\mathcal{M}_{AB\;ik} = \epsilon_{AB}\, g_{l^\star[i} f^\Lambda_{k]} k^{l^\star}_\Lambda + \frac{1}{2} \mathrm{i} P_{\Lambda AB}\, \nabla_i f^\Lambda_k \tag{G.9}$$

$$\mathcal{V}(z,\overline{z},q) = g^2 \Big[\Big(g_{i\overline{\jmath}} k^i_\Lambda k^{\overline{\jmath}}_\Sigma + 4 h_{uv} k^u_\Lambda k^v_\Sigma \Big) \overline{L}^\Lambda L^\Sigma + g^{i\overline{\jmath}} f_i^\Lambda f_{\overline{\jmath}}^\Sigma \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma - 3 \overline{L}^\Lambda L^\Sigma \mathcal{P}^x_\Lambda \mathcal{P}^x_\Sigma \Big] . \tag{G.10}$$

The U(1)-Kähler weight of the Fermi fields is

$$P(\psi_A) = P(\lambda^{\overline{\imath}}_A) = P(\zeta_\alpha) = \frac{1}{2} . \tag{G.11}$$

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JHEP03(2002)025